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# 1 **Connectivity-consistent mapping method for 2D discrete fracture networks**

2 Delphine Roubinet, Jean-Raynald de Dreuzy and Philippe Davy

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## 4 **Abstract**

5 We present a new flow computation method in 2D Discrete Fracture Networks (DFN) intermediary  
6 between the classical DFN flow simulation method and the projection onto continuous grids. The  
7 method divides the simulation complexity by solving for flows successively at a local mesh scale and  
8 at the global domain scale. At the mesh scale, flows are determined by classical DFN flow  
9 simulations and approximated by an Equivalent Hydraulic Matrix (EHM) relating heads and flow  
10 rates discretized on the mesh borders. Assembling the Equivalent Hydraulic Matrices provides for a  
11 domain-scale discretization of the flow equation. The Equivalent Hydraulic Matrices transfer the  
12 connectivity and flow structure complexities from the mesh scale to the domain scale. Compared to  
13 existing geometrical mapping or equivalent tensor methods, the EHM method broadens the  
14 simulation range of flow to all types of 2D fracture networks both below and above the  
15 Representative Elementary Volume (REV). Additional computation linked to the derivation of the  
16 mesh-scale Equivalent Hydraulic Matrices increases the accuracy and reliability of the method.  
17 Compared to DFN methods, the EHM method first provides a simpler domain-scale alternative  
18 permeability model. Second, it enhances the simulation capacities to larger fracture networks where  
19 flow discretization on the DFN structure yields system sizes too large to be solved using the most  
20 advanced multigrid and multifrontal methods. We show that the EHM method continuously moves  
21 from the DFN method to the tensor representation as a function of the mesh-scale discretization. The  
22 balance between accuracy and model simplification can be optimally controlled by adjusting the  
23 domain-scale and mesh-scale discretizations.

## 25 **1. Introduction**

26 Fractured media has been classically modeled using either Discrete Fracture Network (DFN) or  
27 Stochastic Continuum (SC) approaches [Neuman, 2005]. Both approaches have their own advantages  
28 and drawbacks [Hsieh, 1998]. First, they differ by their underlying permeability structure and their  
29 capacity of being specified by existing field data [Hsieh, 1998]. The DFN approach easily accounts  
30 for extensive fracture characterization [Cvetkovic et al., 2004; Davy et al., 2006] while the SC  
31 approach copes more consistently with hydraulic data [Ando et al., 2003]. Second, the simulation of  
32 hydraulic processes requires the development of specific methods using the DFN approach whereas  
33 only standard discretization schemes are required with the SC approach. Third, because the SC  
34 approach simplifies the fracture network structure, it is generally less computationally demanding  
35 than the DFN method. Hybrid approaches have been developed to combine the advantages of the  
36 DFN and SC approaches. Most of them use a DFN approach at the onset for building equivalent  
37 heterogeneous continuous models mapping either the smallest fractures [Lee et al., 2001] or all  
38 fractures in the case of the Fracture Continuum Model (FCM) [Botros et al., 2008; Bourbiaux et al.,  
39 1998; Jackson et al., 2002; Reeves et al., 2008; Svensson, 2001]. Fracture Continuum Models aim at  
40 benefiting both from the structure complexity of DFNs and from the simulation and computational  
41 simplicities of continuous media. The objective is often to use the FCM approximation as a basis for  
42 simulating more computationally demanding transient or multiphase flows [Bourbiaux et al., 1998;  
43 Karimi-Fard et al., 2006].

44 The quality of the FCM models critically depends on the derivation of the block-scale permeabilities  
45 from the DFNs, i.e. on the mapping of the fracture network onto the continuum grid. The block is  
46 considered here as the elementary cell of the continuum grid. Block-scale permeabilities are obtained  
47 either from geometrical characteristics [Botros et al., 2008; Svensson, 2001] or through block-scale

48 numerical simulations of flow [*Jackson et al.*, 2002]. Potential errors stem from differences between  
49 the derived scalar or tensor permeabilities and the effective flows within the block. They arise from  
50 the difficulty to account for complex fracture connectivity on a broad range of scales. For mapping  
51 based on geometrical rules, errors decrease with finer discretization whereas for mapping based on  
52 hydraulic computation of the equivalent permeability tensor, errors increase below the  
53 Representative Elementary Volume [*Long et al.*, 1982]. *Jackson et al.* [2002] corrected part of the  
54 latter error by using a larger simulation zone, namely the “guard zone”, designed to remove dummy  
55 additional fracture connectivity with the sides of the block. FCMs keep the general connectivity  
56 structure above the scale of the block but remove most of the connectivity effects at lower scales.  
57 This results in less flow localization at the block scale and in difficulties in defining an equivalent  
58 block permeability tensor. A simple assessment criterion of the relevance of the tensor representation  
59 is the difference between flows on opposite block faces. They are equal in the tensor representation.  
60 Their difference is expected to increase steeply for complex networks below the REV scale as shown  
61 in the example of Figure 1. To avoid handling complex connectivity at the block scale, existing FCM  
62 methods are applied either at scales close to the smallest fractures modeled [*Botros et al.*, 2008;  
63 *Reeves et al.*, 2008] or at scales larger than the Representative Elementary Volume (REV)  
64 [*Durlofsky*, 1991; *Jackson et al.*, 2002]. The first methods, i.e. the methods applicable to scales close  
65 to the smallest fracture modeled, represent permeability by a scalar or a diagonal tensor. They  
66 require fine grids for fractured medium representation but can be highly accurate for not too dense  
67 fracture networks [*Botros et al.*, 2008]. The second methods, i.e. the methods applicable to scales  
68 larger than the REV, represent permeability by an anisotropic full tensor defined by three 2D  
69 parameters  $K_{xx}$ ,  $K_{yy}$  and  $K_{xy}=K_{yx}$ . They require the a priori knowledge of the REV and are hence more  
70 suited to dense fracture networks. Their drawbacks are the strong homogenization of flow, their  
71 applicability to a restricted scale range and the increase of the numerical error with the refinement of  
72 discretization.

73 None of these methods applies between the scale of the smallest fractures modeled and the REV, a  
74 scale range that spans orders of magnitude for multiscale fracture networks (i.e. fracture networks for  
75 which the fracture-length distribution is a power law) [Bonnet *et al.*, 2001; de Dreuzy *et al.*, 2001b].  
76 In fact, this scale range extends at least from the connectivity scale to the REV scale. The  
77 connectivity scale is the scale at which networks are just connected. It ranges from meters to  
78 kilometers [Berkowitz *et al.*, 2000; Davy *et al.*, 2009]. Because of the fracture transmissivity  
79 variability, the REV scale can be one to three orders of magnitude larger than the connectivity scale  
80 [Baghbanan and Jing, 2007; de Dreuzy *et al.*, 2001a; 2002]. Extending at least from the scales  
81 contributing to connectivity to the REV scale, the scale range of fractures contributing to flow covers  
82 several orders of magnitude from the meter to the kilometer scale. For this scale range, the only  
83 available flow simulation method is the DFN method. The DFN flow simulation method, however, is  
84 limited in terms of fracture number and domain size. The limiting step arises when solving the linear  
85 system issued from the flow discretization on the network structure. With traditional system-solving  
86 methods like the conjugate gradient, limitations stemmed from computation time. However, the new  
87 numerical methods like the multifrontal or algebraic multigrid method, as implemented in  
88 UMFPACK [Davis, 2004] and HYPRE [Falgout *et al.*, 2005], are orders of magnitude faster but  
89 require additional memory [de Dreuzy and Erhel, 2002]. Their sole limitation is the computer  
90 memory. As a rule of thumb, they can solve at most a linear system of rank one million in a couple  
91 of minutes on a personal workstation (Pentium Xeon, 3 GHz, 8 Go). Consequently, improving  
92 simulation capacities is not about speeding up the method but about enabling simulations otherwise  
93 impossible because of memory requirements. We will thus look in this paper at the numerical  
94 memory complexity rather than at the numerical time complexity. Our longer-term strategy is to use  
95 parallel computing for performing Monte-Carlo simulations while sequential individual simulations  
96 remain sequential [Erhel *et al.*, 2009]. This ensures scalability and a minimum of parallel computing  
97 implementation.

98 We propose a new FCM method for the scale range where no existing FCM method is applicable.  
99 Like with the previously-cited FCM methods, the objective is to simplify the domain-scale numerical  
100 scheme and computations while keeping the complexity of the DFN structure. The new method  
101 divides the simulation complexity by solving for flows successively at the local block scale and at  
102 the global domain scale. At the block scale, flows are determined by classical DFN flow simulations  
103 and approximated by an Equivalent Hydraulic Matrix (EHM) relating heads and flow rates  
104 discretized on the mesh borders. Assembling the Equivalent Hydraulic Matrices allows for a domain-  
105 scale discretization of the flow equation. The Equivalent Hydraulic Matrices transfer the connectivity  
106 and flow structure complexities from the block scale to the domain scale. The method is similar to  
107 Boundary Element Methods [*Dershowitz and Fidelibus, 1999*] as it relates heads and flow rates on  
108 the block borders. As the Equivalent Hydraulic Matrices are determined at the block scale by DFN  
109 simulations, we show that the method is systematically applicable regardless of the scale, fracture  
110 density and fracture-length and transmissivity distributions. The method accuracy and complexity are  
111 given by the level of discretization of the block borders and of the domain. We call this method the  
112 Equivalent Hydraulic Matrices (EHM) method as heads and flow rates on the block borders are  
113 linearly linked by a matrix representing the block-scale hydraulic properties rather than by a scalar or  
114 a tensor permeability. This article describes the EHM method (section 2), shows its results compared  
115 to existing methods (section 3) and discusses its performance (section 4).

## 116 **2. The Equivalent Hydraulic Matrices method**

117 This section defines the EHM method. Once the domain meshed into elementary blocks, the  
118 principle of the EHM method is to express the block-scale hydraulic properties by a linear  
119 relationship between discretized flow rates and heads on the block borders. This expression will  
120 replace the scalar or tensor models used in classical FCM models. With  $\mathbf{p}_k$  as the discretization  
121 points (also called poles) of the block numbered  $k$ , the vector of flow rates  $\boldsymbol{\phi}_k$  and heads  $\mathbf{H}_k$  on these

122 points are related by the following linear relationship:

$$123 \quad \boldsymbol{\phi}_k = \mathbf{A}_k \cdot \mathbf{H}_k. \quad (1)$$

124 The block matrix  $\mathbf{A}_k$  contains sub-block scale connectivity information and can be considered as the  
125 block-scale constitutive relationship. It is obtained by performing block-scale flow simulations on  
126 the DFN. Once obtained, the block-scale matrices  $\mathbf{A}_k$  are used for simulating flow rates at the system  
127 scale by imposing the continuity of heads and flow rates across the block borders. Relationship (1)  
128 differs a priori from Darcy's law by its relating flow rates to heads and not to head gradients. This is  
129 only a surface difference since the construction method (section 2.2) and the resulting properties of  
130 matrices  $\mathbf{A}_k$  (Appendix A) ensure a dependence of the flow rates on head gradients.

## 131 **2.1. Discretization**

132 Discretization is made up of two parts consisting in discretization of the domain into elementary  
133 blocks (classical meshes) and discretization of block borders into poles. The first discretization  
134 consists in defining the mesh of the Fracture Continuum Model. We use hereafter a regular grid even  
135 though the EHM method can cope with irregular meshes. Each mesh cell will be called a block. The  
136 block contains a subset of the fracture network, i.e. a sub-network, the intersections of which with  
137 the block limits are denoted  $\mathbf{m}_k$ .  $\mathbf{m}_k(i)$  is the  $i^{\text{th}}$  intersection of block  $k$ . The second discretization  
138 consists in splitting up the block borders into segments of constant length  $d_{block}$ , the discretization of  
139 each border starting at the border corner. Each segment contains either zero, one or more than one  
140 fracture border intersection  $\mathbf{m}_k(i)$ . We define poles  $\mathbf{p}_k$  as the centers of those segments containing at  
141 least one intersection (Figure 2). Segments containing no intersection with the subnetwork are  
142 disregarded. The fundamental principle of the EHM method is that all intersections contained in the  
143 same segment are set to the same hydraulic head corresponding to the head of the pole. These  
144 additional equalities reduce the number of unknowns at the cost of the approximation that close

145 intersections have the same hydraulic head. The accuracy of the approximation is function of the  
 146 block discretization ratio  $r_{block}$  defined as the block-border discretization scale  $d_{block}$  normalized by  
 147 the block face length. The coarsest discretization corresponds to  $r_{block}=100\%$  and gives a single pole  
 148 by block face. It leads to a representation close to the tensor representation (Figure 2a). It is,  
 149 however, not equal to a tensor. First, opposite fluxes may not be equal. Second, some faces may not  
 150 be intersected by the network and thus may not have led to a pole. Finer discretizations, obtained for  
 151 decreasing  $r_{block}$  values, lead to more accurate representations converging to the DFN method when  
 152 all poles correspond exactly to one intersetion (Figure 2b). Like in classical numerical methods, we  
 153 will show in section 3 that the numerical error of the EHM method decreases monotonously with the  
 154 block-border discretization ratio  $r_{block}$ , i.e. when shifting from tensor-like to DFN methods.

## 155 2.2. Construction of the block-scale Equivalent Hydraulic Matrices

156 Equivalent Hydraulic Matrix  $\mathbf{A}_k$  expresses the linear relationship between flows and heads on the  
 157 block border discretization. More specifically, by developing relationship (1), coefficient  $\mathbf{A}_k(i, j)$  is  
 158 the contribution of the head at the  $j^{\text{th}}$  pole to the flow at the  $i^{\text{th}}$  pole:

$$159 \quad \phi_k(i) = \sum_{j=1}^{N_p^k} \mathbf{A}_k(i, j) \cdot \mathbf{H}_k(j). \quad (2)$$

160 where  $N_p^k$  is the pole number of block  $k$  and  $\phi_k(i)$  and  $\mathbf{H}_k(i)$  are the flow rate and head,  
 161 respectively, at  $i^{\text{th}}$  pole  $\mathbf{p}_k(i)$ .  $\mathbf{A}_k(i, j)$  is also equal to the flow rate computed at pole  $i$  by imposing a  
 162 fixed head of 1 at pole  $j$  and 0 at the other ones, i.e. a fixed head of 1 for the intersections overlapped  
 163 by the segment centered on pole  $j$  and 0 for the other ones. With these boundary conditions, all  
 164 coefficients of column  $j$  can be simultaneously determined by a single DFN simulation (Figure 3).  
 165 The construction of the full Equivalent Hydraulic Matrix requires  $N_p^k - 1$  simulations and not  $N_p^k$ ,  
 166 since the sum of all elements from a column of  $\mathbf{A}_k$  is equal to zero because of flow conservation

167 (Appendix A). We underline that this method does not require any modification of the fracture  
 168 network structure or any realignment of fractures. The approximation lies exclusively in equating  
 169 flows and heads at the scale of the segment of the border discretization.

### 170 **2.3. Domain-scale flow simulation**

171 Solving the flow equation at the domain scale consists in imposing the continuity of heads and flow  
 172 rates on poles  $\mathbf{p}_k$  positioned on the block faces. External head and flow rate boundary conditions are  
 173 simply implemented by imposing the head in the matrix system for the fixed head values and by  
 174 adding a source term for the fixed flow rates on the corresponding poles, respectively.

175 We note  $P$  the union of all pole points  $\mathbf{p}_k$  with the convention that poles common to two or more  
 176 blocks occur only once in  $P$ .  $P$  is made up of  $N^i$  poles at the interface between two blocks ( $P^i$ ) and of  
 177  $N^f$  poles at the physical limits of the domain ( $P^f$ ). The total number of poles at the domain scale  $N$  is  
 178 equal to the sum of poles of types  $P^i$  and  $P^f$ :

$$179 \quad N = N^i + N^f. \quad (3)$$

180 With  $B(j)$  as the set of blocks sharing pole  $P^i(j)$  and with  $q_{b,P^i(j)}$  as the flow rate at pole  $P^i(j)$  from  
 181 the  $b^{\text{th}}$  block of  $B(j)$ , flow continuity writes:

$$182 \quad \sum_{b \in B(j)} q_{b,P^i(j)} = 0 \quad \forall j \in \llbracket 1, N^i \rrbracket. \quad (4)$$

183 For the  $N^{\text{fd}}$  fixed poles at the domain limit where a Dirichlet boundary condition is applied:

$$184 \quad H^{\text{fd}} = (H^{\text{fd}})_0. \quad (5)$$

185 For the  $N^{\text{fn}}$  poles on the Neumann boundary condition, the imposed flow is simply inserted in

186 equation (4). Equations (1), (4) and (5) lead to a linear system of  $N$  equations of the  $N$  unknown  
187 heads at the poles.

188 The first advantage of the EHM method compared to existing Fracture Continuum Models (FCMs)  
189 is the conservation of connectivity between blocks. In fact, faces intersected by fractures contain at  
190 least one pole whereas faces without intersecting fractures do not have any pole. This prevents  
191 dummy additional connectivity between blocks [Jackson *et al.*, 2002; Reeves *et al.*, 2008]. The  
192 second advantage of the EHM method is the existence of block-scale discretization parameter  $r_{\text{block}}$ ,  
193 which can be used to tune the balance between numerical efficiency and accuracy. The third  
194 advantage of the method is the systematic convergence with discretization and its adjustment to all  
195 kinds of 2D synthetic fracture networks as will be shown in section 3. The main drawbacks of the  
196 EHM method are the necessity to perform block-scale DFN flow simulations and the specificity of  
197 the domain-scale flow simulation that precludes the use of standard softwares like MODFLOW.

### 198 **3. Results**

#### 199 **3.1. Fracture network types**

200 The tested networks have been chosen so that they cover a wide range of networks both above and  
201 below the REV scale, with broad and narrow length and transmissivity distributions (Table 1).  
202 Extreme cases of low and high variability are tested in order to assess the method in highly-  
203 differentiated conditions. Network types include both lattice structures (Table 2.I1) and stochastic  
204 complex fracture networks (Table 2.I2-4). Stochastic fracture networks are characterized by their  
205 density, orientation, length and transmissivity distributions. The domain size given by the ratio of the  
206 domain length to the minimal fracture length is denoted by  $L$  and set to 100. It means that the  
207 fracture length distribution covers two orders of magnitude. Density is fixed by the dimensionless  
208 percolation parameter  $p$ , equal to the sum of the square of the fracture lengths normalized by the

209 domain area.  $p$  is a direct measure of connectivity as it is very close to 5.6 at the percolation  
210 threshold, whatever the other fracture network characteristics [Bour and Davy, 1997]. Three density  
211 values are used for stochastic complex fracture networks and are respectively close to threshold  
212 ( $p=6$ ) and at around two and three times the density at threshold ( $p=10$  and  $p=20$ ). For lattice  
213 structures,  $p$  is close to the number of fractures within the domain and has been chosen equal to 12  
214 and 192 for testing methods on sparse and dense lattices, respectively. Orientations are set to  $0^\circ$  and  
215  $30^\circ$  relative to the main flow directions for the lattice structures and are uniformly distributed for the  
216 complex stochastic fracture networks. For the complex stochastic fracture networks, fracture lengths  
217 are power-law distributed [Bonnet *et al.*, 2001] according to the following distribution function:

$$218 \quad p(l) \sim l^{-a} \quad (6)$$

219 where  $l$  is the fracture length,  $a$  is the characteristic power-law length exponent and  $p(l)$  the fracture  
220 number of length  $l$ . Natural values of  $a$  derived from outcrops range in the interval [2.0,3.5]. Fracture  
221 transmissivity values have been chosen to be either the same for all fractures or broadly distributed  
222 according to a lognormal distribution of logarithmic standard deviation equal to 3 [Tsang *et al.*,  
223 1996]. Flow boundary conditions are classical gradient-like boundary conditions with fixed head on  
224 two opposite domain faces and a constant head gradient on the orthogonal faces (Figure 1a). The  
225 bottom line of Table 2 illustrates the flow distribution computed with a broad transmissivity  
226 distribution and shows the strong channeling induced by the transmissivity distribution.

### 227 **3.2. Comparison criteria**

228 For comparing the performance of the EHM method with other existing methods, we use an accuracy  
229 criterion and a numerical memory complexity criterion. Accuracy is defined as the mean difference  
230 between the inlet and outlet flows and their reference counterparts. The reference is obtained from

231 the direct simulation on the domain-scale discrete fracture network. By denoting  $\Phi_m^{f_i}$  and  $\Phi_{ref}^{f_i}$  the  
 232 flow rates obtained respectively by the method “ $m$ ” and the reference method on face  $f_i$ , the  
 233 comparison criterion writes:

$$234 \quad flow\_error_m = \frac{1}{2} \left( \left| \frac{\Phi_m^{f_l} - \Phi_{ref}^{f_l}}{\Phi_{ref}^{f_l}} \right| + \left| \frac{\Phi_m^{f_r} - \Phi_{ref}^{f_r}}{\Phi_{ref}^{f_r}} \right| \right) \times 100 \quad (7)$$

235 where  $f_l$  and  $f_r$  stand for the left and right vertical domain faces.

236 The memory complexity criterion is taken as the number of non-zero elements  $nnz$  of matrix  $B$  in the  
 237 linear system  $Bx=b$  issued from the discretization of the flow equation at the domain scale. Even if  
 238 the number of non-zero elements is not the ideal criterion, it is still better than the system size in this  
 239 case where the limitation lies rather in memory requirements than in computation time. All results  
 240 represent averages over 10 simulations. We have checked that for the most complex cases  $D0$  and  
 241  $D1$ , 10 and 100 simulations give very close results. Accuracy and numerical memory complexity  
 242 results are computed for several discretizations characterized by the number of blocks (domain-scale  
 243 discretization) and by  $r_{block}$  (block-scale discretization).

### 244 3.3. Results with existing mapping and tensor methods

245 To assess the Equivalent Hydraulic Matrices method, we compare it with other existing methods:  
 246 first with what we call the ANIS\_GEO method representing permeability by a diagonal tensor  
 247 derived from fracture geometrical mapping onto the blocks and used within a finite volume method  
 248 [Botros *et al.*, 2008] and second with what we call the TENSOR\_SIM method representing  
 249 permeability by a full tensor obtained from block-scale DFN flow simulations and used within a  
 250 mixed hybrid finite element framework (Appendix B). For these two methods, the matrix  
 251 permeability is fixed to  $10^{-12}$  m/s. We use these two methods only when they are strictly applicable.

252 From [Botros *et al.*, 2008], the ANIS\_GEO method is applicable only if the ratio of the block length  
253 to the minimal fracture length is lower than 2.5. For the stochastic complex networks (Table 1 B0-  
254 D1), the ratio of the domain size to the minimal fracture length is  $L=100$ , requiring for the  
255 ANIS\_GEO method a domain-scale discretization of at least  $40\times 40$  blocks. As the TENSOR\_SIM  
256 method relies on the full permeability tensor at the block scale, we have determined this parameter  
257 for all studied networks from the block-scale directional permeability plots (Table 3). The method is  
258 applicable only when the directional permeability is close to an ellipse [Long *et al.*, 1982]. It is the  
259 case for networks *A0*, *A2* and *D0* (Table 3). For the other networks, transmissivity and fracture  
260 length distributions display heterogeneities that cannot be represented by a tensor at the scale of the  
261 block.

262 Table 4 shows the flow error as measured by (7) using the ANIS\_GEO, TENSOR\_SIM and EHM  
263 methods for several domain discretizations. With the ANIS\_GEO method, the flow error decreases  
264 systematically from a  $50\times 50$  to a  $200\times 200$  domain discretization. ANIS\_GEO is particularly accurate  
265 for sparse flow structures (networks with a small fracture density or with a broad transmissivity  
266 distribution). In fact, the simple summation of the fracture contributions induced by the mapping  
267 increases sub-block-scale connectivity and hence increases flow errors. Results also show that  
268 ANIS\_GEO is not applicable to networks with connectivity driven by small fractures ( $3 < a < 3.5$ ),  
269 yielding errors systematically larger than 41%. To be applied systematically, the geometrical  
270 projection method ANIS\_GEO requires high levels of discretization involving large linear systems  
271 (Table 5). Such discretization levels can be achieved in 2D but likely not in 3D.

272 The TENSOR\_SIM method is accurate for regular and dense structures with an error lower than 1%  
273 for network *A0* (Table 4). As opposed to the ANIS\_GEO method, the error decreases when the block  
274 scale increases since the block becomes closer and eventually larger than the REV [Li *et al.*, 2009].  
275 The main drawback of this method is its highly limited range of application. Most of the tested

276 networks of Table 1 did not fulfill its conditions of application.

### 277 **3.4. Assessment of the EHM method**

278 We have tested two levels of block-scale discretization of the EHM method:  $r_{block}=10\%$  (called the  
279 most accurate method) and  $r_{block}=25\%$  (called the least accurate method). The EHM method gives  
280 much smaller errors than those given by the geometrical and tensor methods ANIS\_GEO and  
281 TENSOR\_SIM (Table 4) except for  $A0$  (dense lattice structure with uniform fracture transmissivity)  
282 and  $D0$  (dense fracture network with uniform fracture transmissivity) with a domain discretized by  
283  $10\times 10$  blocks and  $r_{block}=25\%$ . For these two cases, the tensor method gives smaller errors than the  
284 least accurate EHM method. In fact, the tensor method is very accurate because the REV is smaller  
285 than the block. The large errors of the least accurate EHM method are linked to the large number of  
286 fracture intersection points with the block border set to the same head, i.e. the head of the  
287 corresponding pole. The merged points are quantified by the border merging percentage  $p_{border}$  equal  
288 to the difference in percentage between the intersection point and pole numbers.  $p_{border}$  is 0% in the  
289 absence of any approximation of the block-scale discretization and increases as larger  
290 approximations are induced by the use of a smaller number of poles for the block-scale  
291 discretization. For  $A0$  and  $D0$  with the  $10\times 10$  domain discretization and  $r_{block}=25\%$ ,  $p_{border}$  is larger  
292 than 90% and 70%, respectively (Table 6). This explains the cases where the EHM method is less  
293 accurate than the TENSOR\_SIM method. For the same networks with finer domain discretizations  
294 ( $30\times 30$  and  $50\times 50$  blocks), trends are reversed and the EHM method becomes more accurate than  
295 the tensor method. For lattice cases, the flow error with the EHM method is smaller than 5% for a  
296 domain discretization of  $50\times 50$  blocks.

297 For stochastic complex fracture networks, flow errors range from 0.11% to 180% with a majority of  
298 errors below 10% (Table 4). Errors larger than 10% affect cases  $B2$  and  $C2$  characterized by a coarse

299 discretization of  $10 \times 10$  blocks and by networks with the narrowest length distribution corresponding  
300 to  $a=3.5$ . The latter fracture networks have the largest number of fractures and fracture border  
301 intersections inducing first a stronger decrease in the numerical memory complexity (Table 5), and  
302 then larger values of point merging percentages  $p_{\text{border}}$  (Table 6). In all other cases, the flow error is  
303 smaller than 5% for a domain discretization of  $50 \times 50$  blocks. With the most accurate method  
304 corresponding to  $r_{\text{block}}=10\%$  and a domain discretization of  $50 \times 50$  blocks, errors range between  
305 0.11% and 2.1%. For 9 out of the 12 test cases for which  $\sigma_{\ln T} = 3$  corresponds to a fracture  
306 transmissivity distribution spanning at least 3 orders of magnitude, errors remain as low as a few  
307 percents showing the very good performance of the EHM method for complex flow structures.

308 Results of Table 4 show two interesting properties of the EHM method. First, errors are not sensitive  
309 to the fracture transmissivity distribution as shown by the comparison of the *D0* and *D1* cases.  
310 Second, errors systematically decrease both with the domain discretization at constant  $r_{\text{block}}$  and with  
311  $r_{\text{block}}$  at constant domain discretization for all complex stochastic fracture networks. These properties  
312 offer possibilities to control the error by decreasing either the domain-scale discretization in blocks  
313 or the block-scale discretization ratio  $r_{\text{block}}$ . We note that all the above simulations have been  
314 performed on the backbone. However the applicability of the EHM method is not restricted to the  
315 backbone as shown by its good performance on infinite clusters (Table 7). Even if errors increase by  
316 a factor of 5 from the backbone to the infinite cluster, they still remain lower than 10% with the least  
317 accurate method ( $r_{\text{block}}=10\%$ ) and a domain discretization of  $50 \times 50$ .

### 318 **3.5. Flow error versus numerical memory complexity**

319 Numerical memory complexity is taken as the number of non-zero elements in the domain-scale  
320 linear system issued from the discretization of the flow equation (*nnz*) (Table 5). *nnz* determines the  
321 memory required to solve the linear system. It does not, however, take into account the computation

322 of the Equivalent Hydraulic Matrices at the block scale as they are not critical in terms of system size  
323 and memory requirements. With the classical ANIS\_GEO and TENSOR\_SIM methods, the  
324 numerical memory complexity increases quadratically with the discretization ratio. With the EHM  
325 method, the numerical memory complexity is more variable and increases more slowly. Whatever  
326 the domain discretization and the value of  $r_{block}$  for complex stochastic fracture networks, EHM  
327 methods yield smaller numerical memory complexity than the DFN method except for the *B0* case.  
328 In the latter case, the proportion of blocks crossed by a single fracture increases the numerical  
329 memory complexity without improving the accuracy.

330 A more advanced evaluation of the methods is proposed by comparing their error according to their  
331 numerical memory complexity (Figures 4-6). For lattice structures (Figure 4 except magenta  
332 symbols), the EHM method is orders of magnitude more accurate than the classical methods at  
333 comparable complexities except for the *A0* case already discussed in section 3.4. Figure 4 also shows  
334 that the accuracy of the TENSOR\_SIM method increases with the numerical memory complexity as  
335 discussed in section 3.3. For the dense complex stochastic fracture network of case *D0* (Figure 4,  
336 magenta symbols), the error with the TENSOR\_SIM method is smaller than the error with all other  
337 methods at very low complexity (11%) but cannot be made smaller by refining the discretization. By  
338 contrast, with the EHM method, the error is larger at small complexity but decreases to less than 1%  
339 for the highest complexities. For the stochastic complex fracture networks (Figures 5-6), errors with  
340 the EHM method decrease with the numerical memory complexity ( $nnz$ ), with a systematic trend  
341 close to  $nnz^{-1}$ . Figures 4-6 show that the errors using the EHM method with  $r_{block}=10\%$  and  $r_{block}$   
342  $=25\%$  are roughly parallel in log-log plots. For the same level of error corresponding to horizontal  
343 lines in Figures 4-6, the  $r_{block}=10\%$  method yields smaller numerical memory complexities than the  
344 method with  $r_{block}=25\%$ .

### 345 3.6. Parameter optimization

346 The choice of the optimal method parameters depends on the targeted accuracy, available  
347 computation time and memory and on the fracture network structure. We illustrate the methodology  
348 to determine the appropriate parameter values on the most complex fracture network presented  
349 before *DI*. Basically, we show in this section that the accuracy is controlled by the discretization  
350 ratio  $r_{block}$  times the length of the block edge while computation time and memory requirements are  
351 controlled by the inverse of the discretization ratio ( $1/r_{block}$ ). The approximation of the method is  
352 performed on the block-border discretization by equating the head of points belonging to the same  
353 discretization segment. The sole parameter influencing accuracy is thus the normalized segment  
354 length  $d_{block}$  equal to the discretization ratio  $r_{block}$  times the length of the block edge divided by the  
355 minimal fracture length. The error *error\_flow* defined in (7) increases monotonously with  $d_{block}$   
356 (Figure 7). Flow errors smaller than 20% are obtained for  $d_{block}$  values smaller than 2. Once the  
357 segment length has been fixed by the targeted accuracy, the computation time and memory  
358 requirements are adjusted by choosing the discretization of the system in blocks controlled by the  
359 parameter  $1/r_{block}$  (Figure 8). Here the computation time refers to the full time of the flow simulation  
360 including the determination of the Equivalent Hydraulic Matrices and the solution of the large  
361 system issued by the domain-scale flow discretization. Memory requirements are still taken as the  
362 number of non-zero elements in the domain-scale matrix (*nnz*). As previously said, *nnz* decreases for  
363 coarser domain discretizations. The computation is mainly controlled by the determination of the  
364 Equivalent Hydraulic Matrices. It first sharply decreases with  $1/r_{block}$  and then increases slightly. The  
365 minimum expresses an optimal distribution of computations between the domain scale and the block  
366 scale. Smaller  $1/r_{block}$  values yield more numerous smaller blocks and more Equivalent Hydraulic  
367 Matrices to determine and in turn an increase of the full computation time by more than order of  
368 magnitude. Large  $1/r_{block}$  values yield less numerous larger blocks which Equivalent Hydraulic  
369 Matrices take a much larger time to determine, increasing the full computation time by at least 50%.

370 Similar results showing the existence of the minimum have been obtained for greater number of  
371 Monte-Carlo simulations and for different fracture network structures.

#### 372 **4. Discussion**

373 The principle of the Equivalent Hydraulic Matrices method is to distribute the numerical complexity  
374 among two scales, the block-scale and the domain-scale. This method introduces a reduction of the  
375 domain-scale numerical memory complexity by coarsening the block-border discretization. The  
376 approximation consists in equating heads on nearby network points. It remains local and adjusts  
377 automatically to the specific network configuration. Like the tensor and geometrical mapping  
378 methods, the EHM method increases connectivity along block interfaces but only through the  
379 introduction of shortcuts between existing paths and not through the connection of otherwise  
380 disconnected faces. Moreover, the connectivity increase is limited to the block borders and does not  
381 affect the connectivity within the block.

382 The EHM method is structured around the block-scale Equivalent Hydraulic Matrices, which transfer  
383 the local connectivity information from the block scale to the domain scale. The Equivalent  
384 Hydraulic Matrices are determined by the configurations of the fracture network within the blocks  
385 but do not depend on the boundary conditions. In other words, the matrices are not intrinsic medium  
386 properties like a tensor but can be used instead of the discrete fracture network in all flow contexts  
387 both above and below the Representative Elementary Volume (REV). The Equivalent Hydraulic  
388 Matrices method is still applicable below the REV due to the adjustment of the block-scale matrices  
389 to the specificity of the connectivity structures.

390 Because the Equivalent Hydraulic Matrices are derived from DFN computations, it is not surprising  
391 that they contain more information than the geometrical projection methods and lead to better  
392 performance at equivalent domain-scale numerical memory complexity. We express the domain-

393 scale numerical memory complexity by the number of non-zero elements ( $nnz$ ) of the linear system  
394 issued from the discretization of the flow equation.  $nnz$  is two to four orders magnitude smaller with  
395 the EHM method than with geometrical projection methods. The EHM method also displays  
396 systematically decreasing flow errors with the domain discretization and block-scale discretization  
397 parameter  $r_{block}$ . This offers possibilities to find the best optimal complexity for a given error  
398 requirement. As seen in section 3.3, this is not possible with the tensor method TENSOR\_SIM and it  
399 requires too fine a domain discretization with the geometrical method ANIS\_GEO.

400 The EHM method is intermediary between the full DFN flow simulation and the tensor method. Like  
401 in the classical tensor methods [Jackson *et al.*, 2002], the method relies on block-scale DFN  
402 simulations. It is also similar to classical numerical methods from several respects. First, it expresses  
403 the relationship between flows and heads on the block borders like many numerical methods such as  
404 finite element or boundary element methods. Second, it converges to the full DFN solution when the  
405 domain discretization or the block-scale discretization increases. As a two-scale method, it shares  
406 similarities with multiscale methods like multigrid methods. It is, however, a pure bottom-up  
407 approach in the sense that the block-scale information is used at domain scale but not the other way  
408 around. From this respect, it is closer to the principle of the multiscale finite element methods  
409 [Efendiev and Hou, 2007] than to the principle of multigrid methods [Wesseling, 2004]. Finally, it  
410 remains opposed to homogenization methods since the Equivalent Hydraulic Matrices strongly  
411 depend on the block-scale fracture network structure and cannot be extrapolated to other blocks or  
412 other scales.

413 However, EHM methods have two drawbacks, the first one being the specificity of the domain-scale  
414 simulation method that precludes the use of commonly available continuous flow simulation  
415 softwares like MODFLOW. The second drawback is the additional numerical time complexity  
416 arising from the computation of the block-scale equivalent matrices. The total numerical complexity

417 includes the solution of the domain-scale linear system and the computation of the Equivalent  
418 Hydraulic Matrices at the block scale. The first contribution is evaluated by the number of non-zero  
419 elements in the domain-scale linear system  $nnz$  used in the previous section. The second contribution  
420 is a function of the number of block-scale simulations multiplied by the complexity of the block-  
421 scale simulations. We have chosen to retain only the first contribution to the numerical complexity  
422 for the two following reasons. First, the complexity of the domain-scale linear system is a critical  
423 constraint. Very large systems corresponding to  $nnz > 10^7$  require parallel computation. While this  
424 constraint is met only for very large systems in 2D, it is current for 3D fracture networks at much  
425 smaller domain scales. Second, the EHM methods will likely be interesting for transient simulations.  
426 In fact, the computation of the EHMs will be performed only once and the complexity of the  
427 transient simulations will depend only on the domain-scale linear system complexity. The choice of  
428 both the domain discretization and the block-scale discretization parameter will be dictated by the  
429 numerical optimization, the performance of simulations through block-scale and domain-scale  
430 computations restricted to manageable sizes, and last but not least by the required accuracy.

## 431 **5. Conclusion**

432 We have presented a new mapping method for solving the flow equation in 2D discrete fracture  
433 networks. The method consists in superposing a mesh onto the fracture network and finding the  
434 relationship between heads and flows on the borders of each block of the mesh. The relationship is  
435 linear and can be expressed in matrix form, hence the name the “Equivalent Hydraulic Matrices”  
436 (EHM) method. We have shown that this linear relationship is fundamentally analog to Darcy’s law  
437 as it is equivalent to relating flows to well-chosen head gradients on block borders. The matrix  
438 coefficients can be determined by block-scale numerical simulations and express equivalent block-  
439 scale permeability between block border zones. The zones are chosen independently for each block  
440 interface and correspond to the discretization of intersection points between the fracture network and

441 the block border. The method is parameterized both by the block-scale discretization parameter  
442 (block-scale discretization distance divided by the characteristic block scale) and the domain  
443 discretization (the domain scale divided by characteristic block scale in each direction). The flow  
444 simulation at the domain scale is performed simply by assembling the block-scale Equivalent  
445 Hydraulic Matrices through head and flow continuity conditions.

446 The interest of the EHM method is to keep good approximations of both the internal block and inter-  
447 block connectivities. Discretization is performed at a local scale and adjusts automatically to local  
448 fracture network configurations. We show on a broad range of 2D fracture networks with different  
449 density, fracture length and transmissivity distributions that the relative error of the method decreases  
450 systematically with the domain discretization and the block-scale discretization parameter, allowing  
451 for a possible automatic control of the method accuracy. We also show that the relative error of the  
452 EHM method remains restricted to a few percents for a coarse domain discretization ( $30 \times 30$  to  
453  $50 \times 50$ ), whatever the network geometrical structure and the fracture transmissivity distribution. The  
454 main advantage is its applicability to all kind of network structures, whereas the tensor method can  
455 only be used for blocks larger than the Representative Elementary Volume, a too restrictive  
456 condition for general DFN simulations. Geometrical methods give results of comparable accuracy  
457 for a much larger domain discretization leading to domain-scale numerical memory complexities  
458 orders of magnitude larger than the numerical memory complexity of the EHM method. The EHM  
459 method enables large-scale 2D flow simulation networks. We intend to test its performance on 3D  
460 fracture network simulations and in transient flow contexts.

461 **Appendices**

462 *Appendix A: Property of the Equivalent Hydraulic Matrix*

463 With the construction method described in section 2.2,  $\mathbf{A}_k$  has several properties. First, by imposing  
464 a fixed head of 1 at pole  $j$  and 0 at the other ones as boundary conditions, the flow goes into the  
465 block by  $\mathbf{p}_k(j)$  and outward through the other poles  $\mathbf{p}_k(i)$  ( $i \neq j$ ). Considering the flow going into  
466 the block as positive and the flow going outward as negative leads to:

467 
$$\begin{cases} \mathbf{A}_k(j, j) \geq 0 \\ \mathbf{A}_k(i, j) \leq 0, i \neq j \end{cases} \quad (8)$$

468 Second, for a given column  $j$ , all elements  $\mathbf{A}_k(i, j)$  are determined simultaneously by solving the  
469 flow equation; mass conservation implies that

470 
$$\sum_{i=1}^{N_p^k} \mathbf{A}_k(i, j) = 0. \quad (9)$$

471 Or similarly:

472 
$$\mathbf{A}_k(i, i) = -\sum_{j=1, j \neq i}^{N_p^k} \mathbf{A}_k(j, i). \quad (10)$$

473 Third, because the reciprocity principle is applicable in the case of Darcian flow [Barker, 1991],  $\mathbf{A}_k$   
474 is symmetric:

475 
$$\mathbf{A}_k(i, j) = \mathbf{A}_k(j, i). \quad (11)$$

476 Fourth, we show that the linear relationship (1) between flows and heads with property (10) leads to  
477 a relationship between flows and head gradients. In fact:

$$\phi_k(i) = \sum_{j=1}^{N_p^k} \mathbf{A}_k(i, j) \times \mathbf{H}_k(j)$$

$$\phi_k(i) = \sum_{j=1, j \neq i}^{N_p^k} \mathbf{A}_k(i, j) \times \mathbf{H}_k(j) + \mathbf{A}_k(i, i) \times \mathbf{H}_k(i)$$

478 and using (10):

$$\phi_k(i) = \sum_{j=1, j \neq i}^{N_p^k} \mathbf{A}_k(i, j) \times (\mathbf{H}_k(j) - \mathbf{H}_k(i))$$

$$479 \quad \phi_k(i) = \sum_{j=1}^{N_p^k} \mathbf{A}_k(i, j) \times x_{k,ij} \times \frac{(\mathbf{H}_k(j) - \mathbf{H}_k(i))}{x_{k,ij}} \quad (12)$$

480 where  $x_{k,ij}$  is the distance between poles  $\mathbf{p}_k(i)$  and  $\mathbf{p}_k(j)$ . Equation (12) shows that flow  $\phi_k(i)$  at  
 481  $\mathbf{p}_k(i)$  is the sum of the head gradients from  $\mathbf{p}_k(i)$  to the other poles. Equation (12) gives a simple  
 482 interpretation of  $\mathbf{A}_k(i, j) \times x_{k,ij}$ .  $\mathbf{A}_k(i, j) \times x_{k,ij}$  is the proportionality coefficient between flow  
 483  $\phi_k(i)$  and the head gradient  $(\mathbf{H}_k(j) - \mathbf{H}_k(i))/x_{k,ij}$  between  $\mathbf{p}_k(i)$  and  $\mathbf{p}_k(j)$ .  $\mathbf{A}_k(i, j) \times x_{k,ij}$  can  
 484 thus be interpreted as an “equivalent transmissivity” between the  $i^{\text{th}}$  and  $j^{\text{th}}$  poles.

485

486 *Appendix B: Tensor permeability and finite elements (TENSOR\_SIM method)*

487 The Equivalent Hydraulic Matrices method consists in dividing the domain into blocks and  
488 describing block-scale hydraulic properties using Equivalent Hydraulic Matrices. The discretization  
489 of the block borders by poles  $\mathbf{p}_k$  is determined by the block-scale discretization parameter  $r_{block}$ ,  
490 which is the ratio of the block-scale distance discretization to the block length. This parameter  $r_{block}$   
491 drives the discretization of intersections  $\mathbf{m}_k$  between block borders and fractures. Coefficients of the  
492 EHMs are determined by simulations at the block scale as described in section 2.2. The EHMs are  
493 equivalent to tensors in that they impose the following discretization and construction rules: (1)  
494  $r_{block}=100\%$ , i.e. each block border is discretized by at most one pole, (2) each block border is  
495 represented by one pole (even if there is no intersection point), (3) matrix coefficients are determined  
496 by applying head gradient boundary conditions in the vertical and horizontal directions [Renard *et*  
497 *al.*, 2001], (4) the computed flow rates used for the determination of the coefficients are the  
498 directional flow rates, i.e. the mean of the flow rates going out of the domain through borders  
499 perpendicular to the studied direction, and (5) coefficients are corrected to obtain symmetric positive  
500 definite tensors [Long *et al.*, 1982]. Adding these rules of determination, the Equivalent Hydraulic  
501 Matrices become tensors that describe block-scale permeability. Computed block-scale tensors are  
502 used within a classical mixed hybrid method adapted for quadrangles to simulate flow at the domain  
503 scale [Chavent and Roberts, 1991]. We denote this method the TENSOR\_SIM method.

504

505 **Notations**

- 506  $K_{xx}$  permeability in the  $x$ -direction due to a head gradient in the  $x$ -direction, m/s.
- 507  $K_{yy}$  permeability in the  $y$ -direction due to a head gradient in the  $y$ -direction, m/s.
- 508  $K_{xy}$  permeability in the  $x$ -direction due to a head gradient in the  $y$ -direction, m/s.
- 509  $K_{yx}$  permeability in the  $y$ -direction due to a head gradient in the  $x$ -direction, m/s.
- 510  $\mathbf{p}_k$  vector of poles.
- 511  $\Phi_k$  vector of flow rates at the poles for block  $k$ , m<sup>2</sup>/s.
- 512  $\mathbf{H}_k$  vector of heads at the poles for block  $k$ , m.
- 513  $\mathbf{A}_k$  Equivalent Hydraulic Matrix of block  $k$ , m/s.
- 514  $\mathbf{m}_k$  vector of intersections between the fractures and the faces of block  $k$ .
- 515  $d_{block}$  discretization distance of block borders, m.
- 516  $r_{block}$  discretization ratio of block borders.
- 517  $N_p^k$  number of poles of block  $k$ .
- 518  $x_{k,ij}$  distance between the  $i^{\text{th}}$  and  $j^{\text{th}}$  poles, m.
- 519  $P$  union of all poles.

520	$P^i$	union of poles on block interfaces.
521	$P^f$	union of poles on domain faces.
522	$N$	total number of poles.
523	$N^i$	number of poles of type $P^i$ .
524	$N^f$	number of poles of type $P^f$ .
525	$B(j)$	set of blocks sharing pole $P^i(j)$
526	$q_{b,P^i(j)}$	flow rate at poles $P^i(j)$ from the $b^{\text{th}}$ block, $\text{m}^2/\text{s}$ .
527	$N^{\text{fd}}$	number of poles on the Dirichlet boundary condition.
528	$N^{\text{fn}}$	number of poles on the Neumann boundary condition.
529	$H^{\text{fd}}$	head of poles on the Neumann boundary condition, m
530	$(H^{\text{fd}})_0$	fixed head on the Neumann boundary condition, m
531	$p$	percolation parameter.
532	$l$	fracture length, m.
533	$p(l)$	fracture length distribution.
534	$a$	power law exponent.

- 535  $\Phi_m^{f_i}$  flow rate computed by the method “ $m$ ” on the face  $f_i$ ,  $\text{m}^2/\text{s}$ .
- 536  $\Phi_{ref}^{f_i}$  flow rate computed by the reference method on the face  $f_i$ ,  $\text{m}^2/\text{s}$ .
- 537  $nnz$  number of non-zero elements of the domain-scale linear system.
- 538  $p_{border}$  border discretization percentage

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621

622 **Figure captions**

623 Figure 1 – Fracture network at the block scale (a) and corresponding flows (b) for the gradient head  
624 boundary conditions illustrated in (a). Fracture network parameters are the system size  $L$  normalized  
625 by the smallest fracture length ( $L = 100$ ), the fracture density number twice larger as its value at  
626 percolation threshold, the power-law fracture length exponent of 2.5 and the lognormal  
627 transmissivity distribution of logarithmic standard deviation 3.0. Boundary flows integrated on the  
628 domain sides and normalized by the mean fracture transmissivity are given in (b). They display large  
629 differences between opposite sides and illustrate the non-tensor nature of the flows.

630 Figure 2 – Principle of the block-border discretization with two different discretization scales  $d_{\text{block}}$   
631 corresponding to the side length (a) and to half of it (b). The backbone of the sub-network contained  
632 in the block is represented by the grey segments. Intersections  $m_k$  between the backbone and the  
633 block borders are the black dots. Discretization segments and poles  $p_k$  are respectively the color  
634 dashed segments and crosses. In (a), the four discretization segments intersect the backbone in one or  
635 two points. The four poles corresponding to the four crosses are thus defined and the Equivalent  
636 Hydraulic Matrix (EHM) is of rank 4. In (b), only six of the eight discretization segments intersect  
637 the backbone leading to the definition of 6 poles and to an EHM of rank 6.

638 Figure 3 – Principle of the determination of one of the columns of the Equivalent Hydraulic Matrix  
639  $A_k$ . In this example, block  $k$  is made up of five intersections between the sub-network and the block  
640 borders (black points) and four poles (blue crosses). The boundary conditions applied to poles  
641 illustrated in (a) are a fixed head of 1 for the 2<sup>nd</sup> pole and 0 for the other ones. They condition the  
642 boundary conditions applied to the intersections illustrated in (b), which are a fixed head of 1 for the

643 intersections represented by the 2<sup>nd</sup> pole and 0 for the other ones. Flow rates in poles (d) are deduced  
644 from flow rates at the intersections (c). The flow rate at the  $i^{\text{th}}$  pole is the sum of the flow rates at the  
645 intersections represented by this pole. The elements of the second column of the matrix  $A_k$  are  
646 deduced from flow rates computed at the poles (e).

647 Figure 4 – *flow\_error* versus numerical memory complexity (*nnz*) for lattice structures and dense  
648 fracture networks with constant fracture transmissivity (magenta symbols). The grey area underlines  
649 a lower part of the graph where errors range between  $5 \times 10^{-6}\%$  and  $10^{-4}\%$ . The dashed horizontal line  
650 pictures the 10% error value. The dashed diagonal lines are power-law functions of exponent -1 and  
651 are meant as a guide for the eye for the decrease tendency of the EHM method. Note that errors  
652 larger than  $10^3$  are not represented.

653 Figure 5 – *flow\_error* versus numerical memory complexity (*nnz*) for stochastic complex fracture  
654 networks at threshold with distributed fracture transmissivities. The dashed horizontal line pictures  
655 the 10% error value. The dashed diagonal lines are power-law functions of exponent -1 and are  
656 meant as a guide for the eye for the decrease tendency of the EHM method. Note that errors larger  
657 than  $10^3$  are not represented.

658 Figure 6 – *flow\_error* versus numerical memory complexity (*nnz*) for stochastic complex fracture  
659 networks with distributed fracture transmissivities. The dashed horizontal line pictures the 10% error  
660 value. The dashed diagonal lines are power-law functions of exponent -1 and are meant as a guide  
661 for the eye for the decrease tendency of the EHM method. Note that errors larger than  $10^3$  are not  
662 represented.

663 Figure 7 – *flow\_error* versus  $d_{block}$  the discretization ratio  $r_{block}$  times the length of the block edge for  
664 the network *DI* (domain size  $L=100$ ).

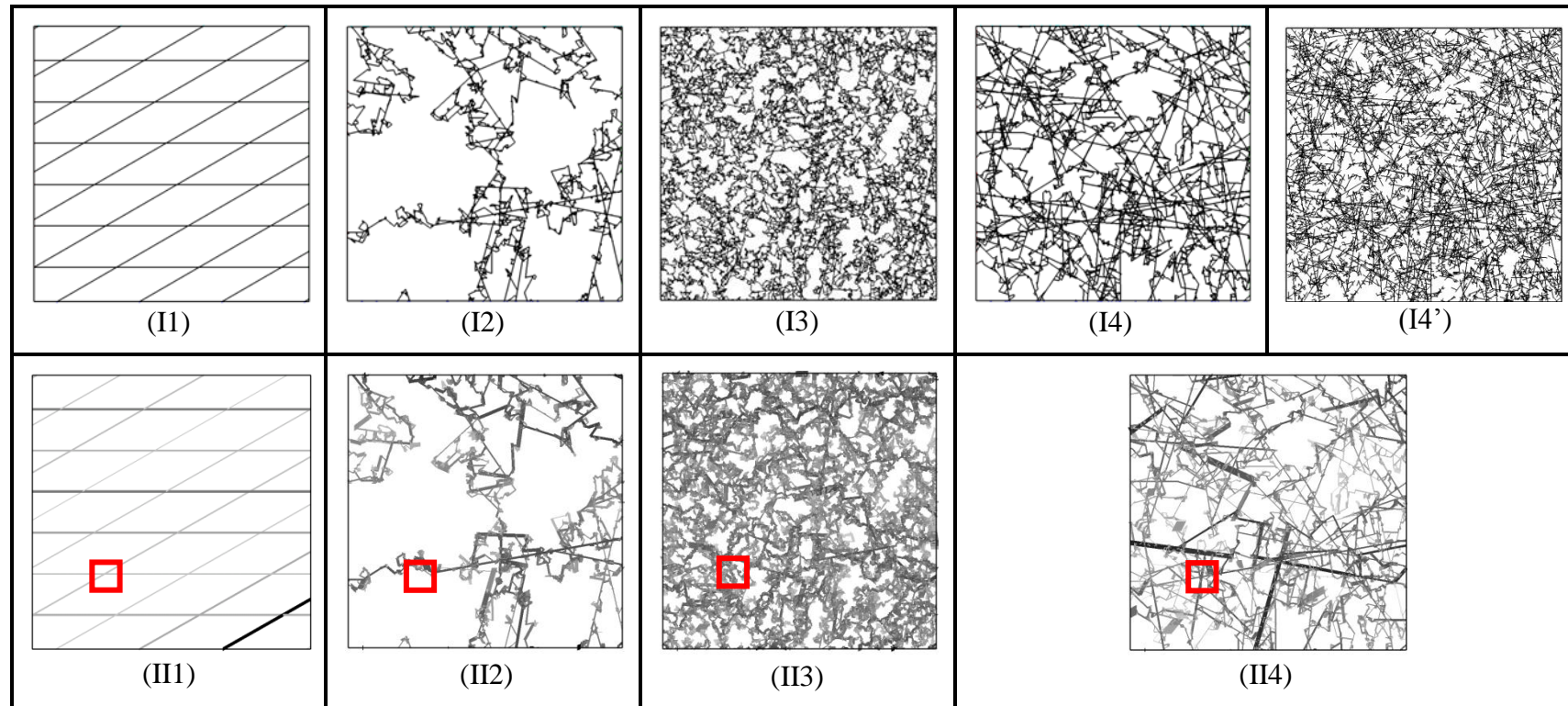
665 Figure 8 – Computation time (red dashed line) and numerical memory complexity taken as the  
666 number of non-zero elements in the largest matrix (black line) as a function of block size divided by  
667 the segment discretization length  $1/r_{block}$  for  $DI$  with  $d_{block}$  equal to 1.

668

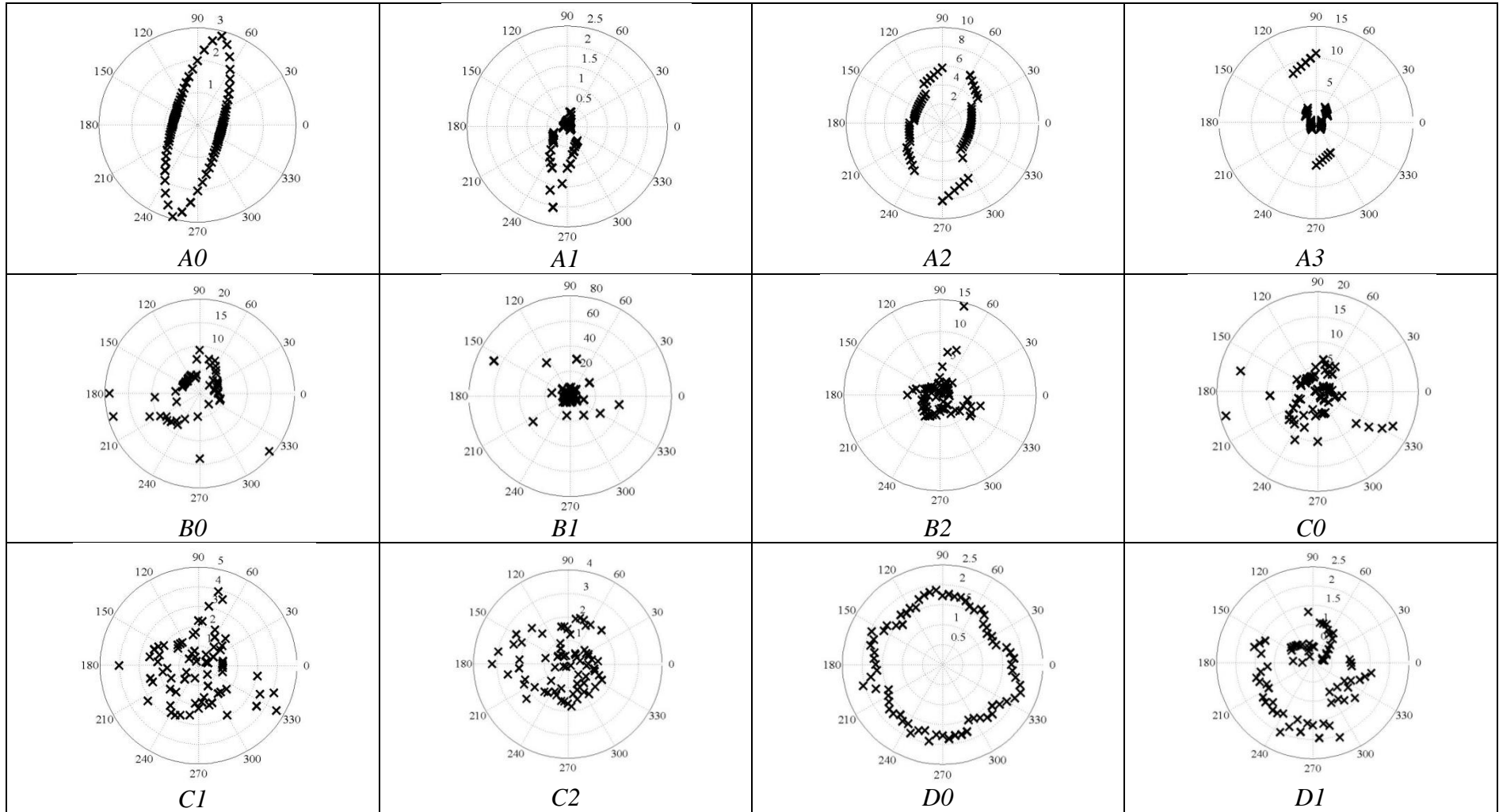
Network type	Parameters				Table 2
	$p$	$a$	$\theta$	$\sigma_{lnT}$	
<i>A0</i>	192	1	0°, 30°	0	
<i>A1</i>	192	1	0°, 30°	3	
<i>A2</i>	12	1	0°, 30°	0	
<i>A3</i>	12	1	0°, 30°	3	I1
<i>B0</i>	6	2.5	Uniform	3	
<i>B1</i>	6	3	Uniform	3	I2
<i>B2</i>	6	3.5	Uniform	3	
<i>C0</i>	10	2.5	Uniform	3	
<i>C1</i>	10	3	Uniform	3	
<i>C2</i>	10	3.5	Uniform	3	I3
<i>D0</i>	20	2.5	Uniform	0	
<i>D1</i>	20	2.5	Uniform	3	I4

670 Table 1 – Characteristics of the tested fracture networks. The first four networks are on-lattice structures whereas the other ones are off-lattice  
671 structures. The ratio of the domain size  $L$  to the length of the smallest fracture  $l_{min}$  is set to 100. The fracture density is characterized by the  
672 percolation parameter  $p$  [Bour and Davy, 1998]. Fractures either cross the whole domain ( $a=1$ ) or are distributed according to a power-law  
673 distribution ( $a>1$ ). Fracture orientations ( $\theta$ ) are either specified to a set of fixed angles (first four cases) or uniformly distributed. Fracture  
674 transmissivity is constant ( $\sigma_{lnT} = 0$ ) or lognormally distributed with a lognormal standard deviation ( $\sigma_{lnT} = 3$ ). The last column indicates the  
675 figure numbers displaying an example of the network type in Table 2. Fracture network types are classified in family of networks: “A” is for  
676 lattice structures, “B” for networks at percolation threshold, “C” for networks with an intermediary fracture density and “D” for dense networks.

677



678 Table 2 – Illustration of the tested fracture networks with examples of backbones (I1-I4), infinite clusters (I4') and flows (II1-II4) obtained with  
679 the gradient boundary conditions illustrated by Figure 1a and computed with a broadly distributed fracture transmissivity  $\sigma_{mT} = 3$  (see Table 1).  
680 Flow magnitude is represented by grey intensity and segment width. I1, I2, I3 and I4-4' correspond to network types *A3*, *B1*, *C2* and *D1* (Table  
681 1). Red squares stand for an elementary block corresponding to a domain discretization of  $10 \times 10$  blocks.



682 Table 3 – Polar plots of permeability for the networks of Table 1, representing the permeability versus the polar angle  $\theta$ . Each point represents  
 683 for a given  $\theta$  the permeability computed in a square of side length  $L/3$  (where  $L$  is the domain size), of axis rotated by  $\theta$  and centered on the  
 684 initial system center.

Domain discretization	ANIS_GEO				TENSOR_SIM			EHM (10%)			EHM (25%)		
	10×10	30×30	50×50	200×200	10×10	30×30	50×50	10×10	30×30	50×50	10×10	30×30	50×50
<i>A0</i>	33	33	31	20	0.14	0.04	0.92	$5.5 \times 10^{-5}$	$2.0 \times 10^{-5}$	$2.7 \times 10^{-5}$	33	$2.3 \times 10^{-5}$	$2.8 \times 10^{-5}$
<i>A1</i>	44	49	47	2.3				$4.9 \times 10^{-5}$	$1.5 \times 10^{-5}$	$3.2 \times 10^{-5}$	25	$1.6 \times 10^{-5}$	$3.3 \times 10^{-5}$
<i>A2</i>	49	2.3	2.0	0.3	7.5	35	37	$3.4 \times 10^{-5}$	$3.1 \times 10^{-5}$	$1.4 \times 10^{-5}$	$3.4 \times 10^{-5}$	$3.2 \times 10^{-5}$	$1.4 \times 10^{-5}$
<i>A3</i>	23	6.0	5.6	1.2				$2.1 \times 10^{-4}$	$6.3 \times 10^{-5}$	$4.4 \times 10^{-5}$	$2.1 \times 10^{-4}$	$6.3 \times 10^{-5}$	$4.4 \times 10^{-5}$
<i>B0</i>			22	4.7				0.42	0.25	0.11	3.1	1.2	0.59
<i>B1</i>			$1.2 \times 10^3$	41				3.6	0.73	0.29	10	4.2	2.6
<i>B2</i>			$3.4 \times 10^4$	$2.2 \times 10^2$				45	1.6	1.5	81	45	4.8
<i>C0</i>			78	49				1.0	0.5	0.2	6.1	1.3	0.87
<i>C1</i>			$2.1 \times 10^3$	93				5.9	1.3	1.5	33	5.7	5.1
<i>C2</i>			$1.2 \times 10^4$	$4.4 \times 10^2$				23	4.5	2.1	$1.8 \times 10^2$	29	13
<i>D0</i>			$1.7 \times 10^2$	28	11	21	50	2.8	0.89	0.51	21	5.1	2.9
<i>D1</i>			$4 \times 10^2$	19				3.8	0.75	0.45	23	5.7	2.8

686 Table 4 – *flow\_error* as defined by equation (7) for the backbone of the fracture network types defined in Table 1 and for the three  
687 computational methods ANIS\_GEO, TENSOR\_SIM and EHM at different discretization levels. Domain discretization refers to the ratio of the  
688 domain size to the block size in each direction. EHM methods are characterized in brackets by their block-scale discretization parameter  $r_{block}$   
689 equal to the ratio expressed in % between the block-scale discretization distance  $d_{block}$  and the block scale. Empty cells mean that the conditions  
690 of application of the method are not fulfilled in the corresponding case.

	DFN	ANIS_GEO				TENSOR_SIM			EHM (10%)			EHM (25%)		
Domain discretization		10×10	30×30	50×50	200×200	10×10	30×30	50×50	10×10	30×30	50×50	10×10	30×30	50×50
<i>A0</i>	26	0.46	4.4	12	200	1.2	12	34	14	110	140	1.2	42	75
<i>A1</i>	26	0.46	4.4	12	200				15	101	140	1.2	41	87
<i>A2</i>	0.16	0.46	4.4	12	200	1.2	12	34	0.7	1.6	2.4	0.6	1.5	2.4
<i>A3</i>	0.16	0.46	4.4	12	200				0.7	1.6	2.4	0.6	1.5	2.4
<i>B0</i>	0.87			12	200				0.45	1.4	2.5	0.3	1.2	2.2
<i>B1</i>	13			12	200				1.1	4.2	6.7	0.6	2.6	4.5
<i>B2</i>	33			12	200				1.5	6.6	12	0.7	3.5	7.1
<i>C0</i>	6.1			12	200				1.6	4.0	6.0	0.9	3.0	5.0
<i>C1</i>	53			12	200				4.5	16	26	1.8	8.6	16
<i>C2</i>	240			12	200				11	56	96	2.8	22	46
<i>D0</i>	52			12	200	1.2	12	34	8.7	27	41	2.9	15	26
<i>D1</i>	51			12	200				8.7	27	41	2.9	15	26

692 Table 5 – Numerical memory complexity expressed as the number of non-zero elements (*nnz*) of the domain-scale linear systems issued from the  
693 flow discretization for the network cases of Table 1. Parameters are identical to those of Table 4. All numbers are expressed in thousands of non-  
694 zero elements.

	EHM (10%)			EHM (25%)		
Domain discretization	10×10	30×30	50×50	10×10	30×30	50×50
<i>A0</i>	70	21	14	91	53	36
<i>A1</i>	69	23	14	91	53	32
<i>A2</i>	4.7	1.7	2.6	13	4.3	2.6
<i>A3</i>	4.7	1.7	2.6	13	4.3	2.6
<i>B0</i>	14	7	5.3	26	14	10
<i>B1</i>	44	23	15	59	37	28
<i>B2</i>	54	32	23	70	49	39
<i>C0</i>	25	12	8.3	41	22	15
<i>C1</i>	53	29	21	70	46	35
<i>C2</i>	72	44	33	86	65	53
<i>D0</i>	50	26	18	70	44	32
<i>D1</i>	50	26	18	70	44	32

696

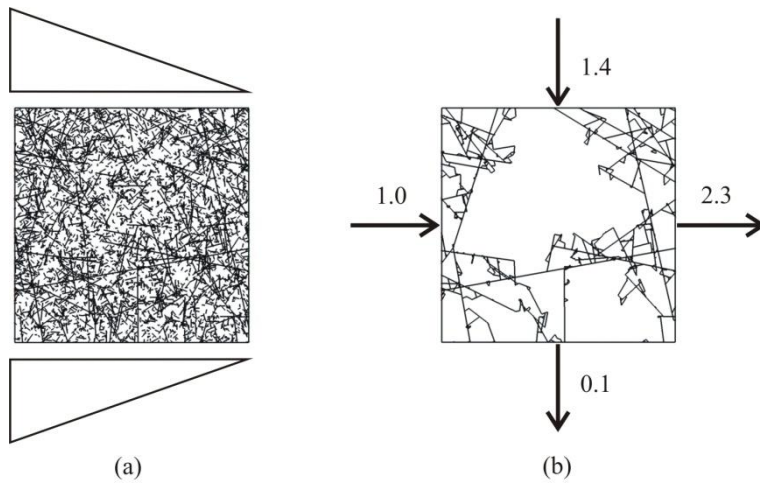
697 Table 6 – Block-scale border merging percentage  $p_{\text{border}}$  for the fracture network types of Table 1.

	ANIS_GEO				TENSOR_SIM			EHM (10%)			EHM (25%)			Table 2
Domain discretization	10×10	30×30	50×50	200×200	10×10	30×30	50×50	10×10	30×30	50×50	10×10	30×30	50×50	
<i>D0</i>			$5.5 \times 10^2$	$1.5 \times 10^2$	37	38	14	11	4.9	3.0	76	23	14	
<i>D1</i>			$1.7 \times 10^3$	$1.5 \times 10^2$				12	4.3	2.4	70	20	11	14'

698

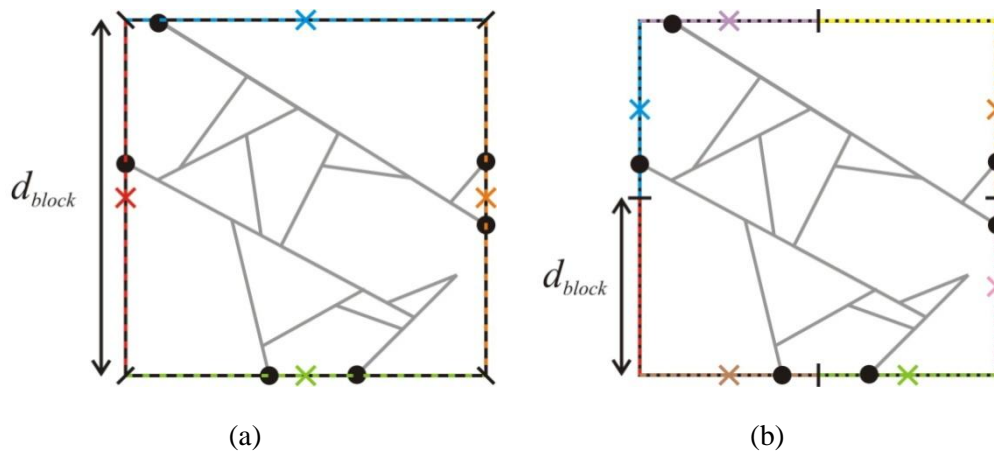
699 Table 7 – *flow\_error* as defined by equation (7) for the infinite cluster of fracture networks *D0* and *D1* (see Table 1 for description). EHM  
700 methods are characterized in brackets by their block-scale discretization parameter  $r_{block}$  equal to the ratio expressed in % of the block-scale  
701 discretization distance  $d_{block}$  to the block scale.

702 **Figures**

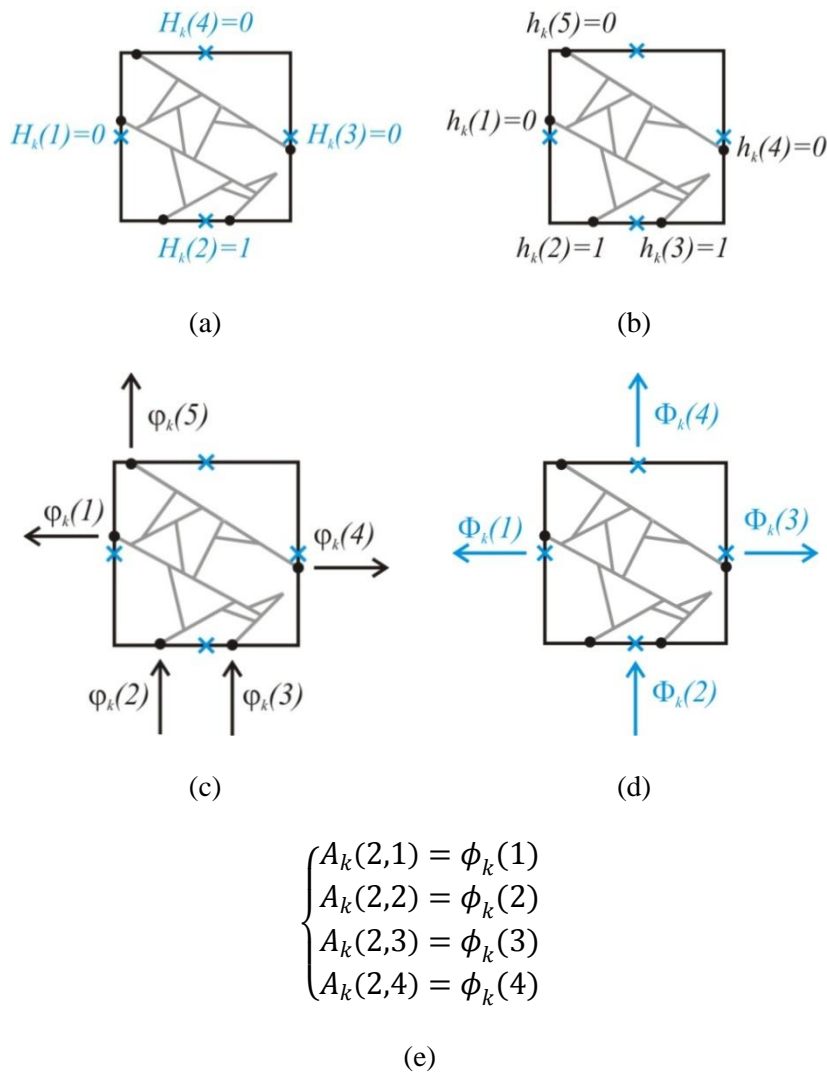


704 Figure 1 – Fracture network at the block scale (a) and corresponding flows (b) for the gradient head  
705 boundary conditions illustrated in (a). Fracture network parameters are the system size  $L$  normalized  
706 by the smallest fracture length ( $L = 100$ ), the fracture density number twice larger as its value at  
707 percolation threshold, the power-law fracture length exponent of 2.5 and the lognormal  
708 transmissivity distribution of logarithmic standard deviation 3.0. Boundary flows integrated on the  
709 domain sides and normalized by the mean fracture transmissivity are given in (b). They display large  
710 differences between opposite sides and illustrate the non-tensor nature of the flows.

711

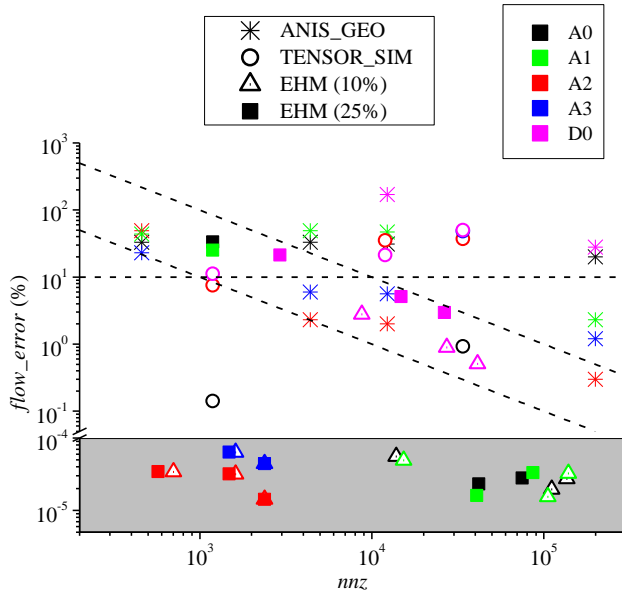


712 Figure 2 – Principle of the block-border discretization with two different discretization scales  $d_{block}$   
 713 corresponding to the side length (a) and to half of it (b). The backbone of the sub-network contained  
 714 in the block is represented by the grey segments. Intersections  $m_k$  between the backbone and the  
 715 block borders are the black dots. Discretization segments and poles  $p_k$  are respectively the color  
 716 dashed segments and crosses. In (a), the four discretization segments intersect the backbone in one or  
 717 two points. The four poles corresponding to the four crosses are thus defined and the Equivalent  
 718 Hydraulic Matrix (EHM) is of rank 4. In (b), only six of the eight discretization segments intersect  
 719 the backbone leading to the definition of 6 poles and to an EHM of rank 6.



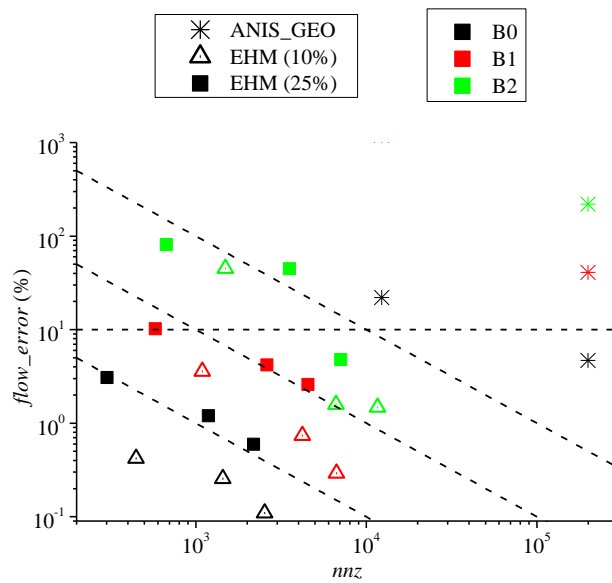
720 Figure 3 – Principle of the determination of one of the columns of the Equivalent Hydraulic Matrix  
 721  $A_k$ . In this example, block  $k$  is made up of five intersections between the sub-network and the block  
 722 borders (black points) and four poles (blue crosses). The boundary conditions applied to poles  
 723 illustrated in (a) are a fixed head of 1 for the 2<sup>nd</sup> pole and 0 for the other ones. They condition the  
 724 boundary conditions applied to the intersections illustrated in (b), which are a fixed head of 1 for the  
 725 intersections represented by the 2<sup>nd</sup> pole and 0 for the other ones. Flow rates in poles (d) are deduced  
 726 from flow rates at the intersections (c). The flow rate at the  $i^{\text{th}}$  pole is the sum of the flow rates at the  
 727 intersections represented by this pole. The elements of the second column of the matrix  $A_k$  are  
 728 deduced from flow rates computed at the poles (e).

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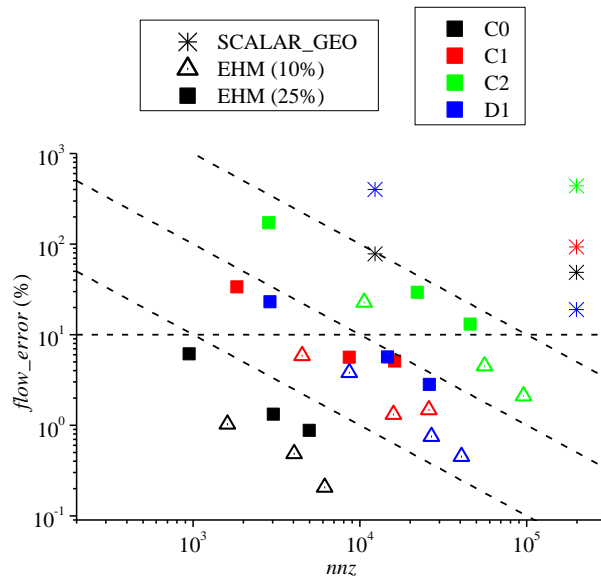
731 Figure 4 –  $flow\_error$  versus numerical memory complexity ( $nnz$ ) for lattice structures and dense  
732 fracture networks with constant fracture transmissivity (magenta symbols). The grey area underlines  
733 a lower part of the graph where errors range between  $5 \times 10^{-6}\%$  and  $10^{-4}\%$ . The dashed horizontal line  
734 pictures the 10% error value. The dashed diagonal lines are power-law functions of exponent -1 and  
735 are meant as a guide for the eye for the decrease tendency of the EHM method. Note that errors  
736 larger than  $10^3$  are not represented.



738

739 Figure 5 –  $flow\_error$  versus numerical memory complexity ( $nnz$ ) for stochastic complex fracture  
 740 networks at threshold with distributed fracture transmissivities. The dashed horizontal line pictures  
 741 the 10% error value. The dashed diagonal lines are power-law functions of exponent -1 and are  
 742 meant as a guide for the eye for the decrease tendency of the EHM method. Note that errors larger  
 743 than  $10^3$  are not represented.

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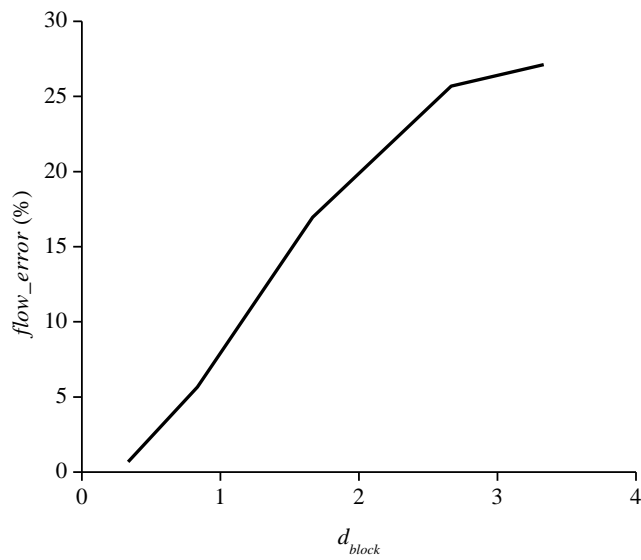


745

746 Figure 6 – *flow\_error* versus numerical memory complexity (*nnz*) for stochastic complex fracture  
747 networks with distributed fracture transmissivities. The dashed horizontal line pictures the 10% error  
748 value. The dashed diagonal lines are power-law functions of exponent -1 and are meant as a guide  
749 for the eye for the decrease tendency of the EHM method. Note that errors larger than  $10^3$  are not  
750 represented.

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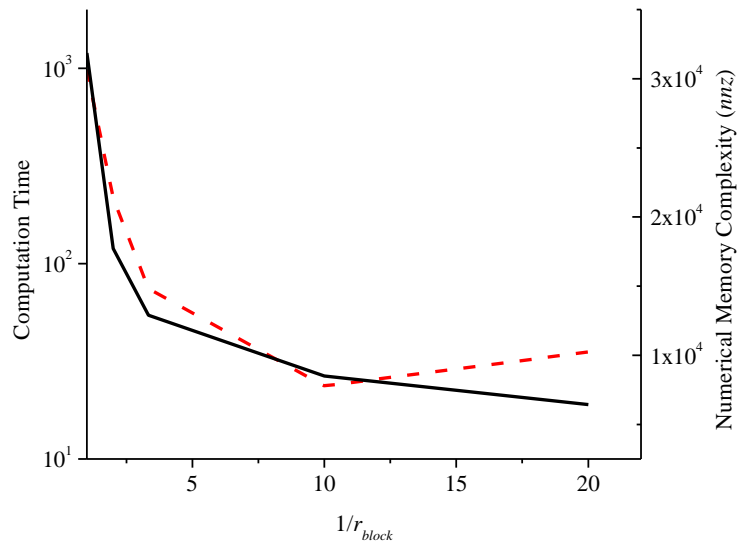


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754 Figure 7  $-flow\_error$  versus  $d_{block}$  the discretization ratio  $r_{block}$  times the length of the block edge for  
755 the network  $DI$  (domain size  $L=100$ ).

756

757



758

759 Figure 8 – Computation time (red dashed line) and numerical memory complexity taken as the  
760 number of non-zero elements in the largest matrix (black line) as a function of block size divided by  
761 the segment discretization length  $1/r_{block}$  for  $DI$  with  $d_{block}$  equal to 1.