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Taking into account truncation problems and geomagnetic model accuracy in assessing computed flows at the core–mantle boundary

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SUMMARY

Since the time Roberts & Scott (1965) first expressed the key 'frozen flux' hypothesis relating the secular variation of the geomagnetic field (SV) to the flow at the core surface, a large number of studies have been devoted to building maps of the flow and inferring its fundamental properties from magnetic observations at the Earth's surface. There are some well-known difficulties in carrying out these studies, such as the one linked to the non-uniqueness of the flow solution [if no additional constraint is imposed on the flow (Backus 1968)] which has been thoroughly investigated. In contrast little investigation has been made up to now to estimate the exact importance of other difficulties, although the different authors are usually well aware of their existence. In this paper we intend to make as systematic as possible a study of the limitations linked to the use of truncated spherical harmonic expansions in the computation of the flow. Our approach does not rely on other assumptions than the frozen flux, the insulating mantle and the large-scale flow assumptions along with some simple statistical assumptions concerning the flow and the Main Field. Our conclusions therefore apply to any (toroidal, steady or tangentially geostrophic) of the flow models that have already been produced; they can be summarized in the following way: first, because of the unavoidable truncation of the spherical harmonic expansion of the Main Field to degree 13, no information will ever be derived for the components of the flow with degree larger than 12; second, one may truncate the spherical harmonic expansion of the flow to degree 12 with only a small impact on the first degrees of the flow. Third, with the data available at the present day, the components of the flow with degree less than 5 are fairly well known whereas those with degree greater than 8 are absolutely unconstrained.

Key words: accuracy, Earth's core surface, fluid flow, geomagnetism, spherical harmonics, truncation.

1 INTRODUCTION

The fluid conducting core of the Earth is where the dynamo effect is taking place. This effect generates the so-called main magnetic field (MF) which we can observe at the Earth's surface and which varies with time. This time variation is called the secular variation (SV). Assuming the mantle is an insulator, it is possible to compute the MF and the SV at the bottom of the mantle by continuing these fields measured at the Earth's surface.

In the fluid conducting core, the induction equation governing the evolution of the magnetic field $\mathbf{B}$ is

$$\partial_t \mathbf{B} = \nabla \times (\mathbf{u} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B} \tag{1}$$

where $\mathbf{u}$ is the flow, ∂_t is the time derivation operator, ∇ the gradient operator and $\eta = (\mu_0 \sigma)^{-1}$ the magnetic diffusivity, with σ the electric conductivity in the core and μ_0 the permeability.

Immediately below the core–mantle boundary (CMB) $u_r = 0$, so that the radial component of equation (1) becomes

$$\partial_t B_r = -\nabla_H \cdot (\mathbf{u} B_r) + \frac{\eta}{r} \nabla^2 (r B_r) \tag{2}$$

where

$$\nabla_H = \nabla - \mathbf{n}\partial_r$$

using spherical coordinates (r, θ, ϕ); $\mathbf{n}$ is the unit outward radial vector, ∂_r is the radial derivation operator, and B_r is the continuous radial component of the MF at the CMB.

Equation (2) shows that the SV is the result of the simultaneous processes of diffusion and advection of the MF at the CMB; it is usually believed that for the time and length scales we are interested in (several tens of years, several hundreds of km at the CMB), the advective process is dominating the production of SV. This leads to the 'frozen flux hypothesis' first introduced by Roberts & Scott (1965). Equation (2) then becomes

$$\partial_t B_r = -\nabla_H \cdot (\mathbf{u}B_r). \tag{3}$$

Equation (3) allows us to extract some information about the fluid flow from magnetic data, the ultimate goal being to derive the pattern of this flow at the CMB. Unfortunately, this is not straightforward.

In fact, some authors believe that the frozen flux hypothesis might break down at some places (Bloxham & Gubbins 1985, 1986, Bloxham, Gubbins & Jackson 1989) which limits the very validity of equation (3) and could affect conclusions on the fluid flow derived from it. But we choose not to discuss this point here (we will do so in another paper), so that we shall assume the frozen flux hypothesis to be true all along this paper.

A second limitation in deriving the flow from equation (3) comes from the well-known non-uniqueness of the solution (Backus 1968). In order to reduce this non-uniqueness one has to make at least one additional assumption on the nature of the flow. This is usually the steady motion hypothesis (Gubbins 1982; Voorhies 1986; Whaler & Clarke 1988), the toroidal hypothesis (Whaler 1980; Bloxham 1989; Lloyd & Gubbins 1990) or the tangentially geostrophic hypothesis (Le Mouël, Gire & Madden 1985; Backus & Le Mouël 1986; Gire & Le Mouël 1990). [See Jault & Le Mouël (1991) and Gubbins (1991) for a recent discussion.] Whether one chooses one assumption rather than another does not matter as far as this paper is concerned (see the remark at the end of Section 5).

These assumptions do not always lead to a practical uniqueness of the flow and it is still necessary (and this is done by all authors) to assume that the flow is large scale. This, at last ensuring a practical uniqueness, will be the third basic assumption (along with the insulating mantle and the frozen flux assumptions) we will make in this paper.

Let us now briefly explain what we intend to do in the following. First recalling why and how the MF, the SV and the motion at the CMB are described by truncated expansions of elementary (MF, SV or motion) fields (Section 2), we underline the unavoidable aspect of the truncation of the expansion of the MF and investigate what kind of limitations this brings about. To do so, we derive some mathematical results concerning the asymptotic behaviour of some specific integrals (the J and J′ interaction integrals, see Section 3) and we make some simple and reasonable statistic assumptions on the behaviours of the MF and of the flow large-degree components (Section 4.1). We are thus able to produce a quantitative estimate for the contribution to the SV of those large-degree terms of the MF and of the flow that are unavoidably neglected because of the truncation of the MF expansion (Sections 4.2 and 4.3). This estimate is also an estimate for the accuracy with which a model of the flow is capable of describing a hypothetically exact SV model. Conversely, it allows us to deduce what part of the flow is constrained by the magnetic data regardless of any observational error (Section 4.4), and therefore allows us to answer our starting question. Taking advantage of the formalism developed in Sections 2 to 4, we also investigate what part of the flow is constrained by the available SV models (now taking into account the errors linked to these) and derive an estimate for the accuracy with which the CMB flow can be calculated from magnetic data (Section 5). Eventually we conclude on several recommendations in order to avoid misunderstandings when looking at a map of a CMB flow model.

2 INTRODUCING TRUNCATED EXPANSIONS

2.1 Expansions of the MF, the SV and the motion at the CMB

We will resume here an algebra that has already been used by many authors but that we need to write down explicitly for the following.

Since the mantle is considered as an insulator, we may write that above the CMB ($r > c$, c being the core radius):

$$\mathbf{B} = -\nabla V \tag{4}$$

where V is an harmonic potential of the form

$$V = c \sum_{n=1}^{\infty} \sum_{m=0}^{n} (b_n^{mc} Y_n^{mc} + b_n^{ms} Y_n^{ms}) \left(\frac{c}{r}\right)^{n+1}.$$

Here we have introduced the spherical harmonics

$$Y_n^{mc} = P_n^m(\cos\theta)\cos m\phi, \qquad Y_n^{ms} = P_n^m(\cos\theta)\sin m\phi,$$

assumed to be Schmidt semi-normalized throughout this paper.

$$\|Y_n^{mc}\| = \|Y_n^{ms}\| = \left[\iint_S (Y_n^{m(c,s)})^2\, dS\right]^{1/2} = \sqrt{\frac{4\pi}{2n+1}}$$

where S is the sphere of radius 1.

At the CMB the radial component of equation (4) leads to

$$B_r = \sum_{n=1}^{\infty} \sum_{m=0}^{n} (n+1)(b_n^{mc} Y_n^{mc} + b_n^{ms} Y_n^{ms}). \tag{5}$$

In very much the same way, and always at the CMB, we can expand the SV radial component into

$$\partial_t B_r = \sum_{n=1}^{\infty} \sum_{m=0}^{n} (n+1)(\dot{b}_n^{mc} Y_n^{mc} + \dot{b}_n^{ms} Y_n^{ms}). \tag{6}$$

The flow $\mathbf{u}$ at the CMB is expanded upon the elementary consoidal and toroidal vector fields given by

$$\mathbf{S}_n^{m(c,s)} = c\nabla_H Y_n^{m(c,s)}, \qquad \mathbf{T}_n^{m(c,s)} = -c\mathbf{n} \times \nabla_H Y_n^{m(c,s)}$$

for which we can define

$$\|\mathbf{S}_n^{m(c,s)}\| = \|\mathbf{T}_n^{m(c,s)}\| = \left[\iint_S (\mathbf{T}_n^{m(c,s)})^2\, dS\right]^{1/2} = \left(n(n+1)\frac{4\pi}{2n+1}\right)^{1/2}, \tag{7}$$

so that $\mathbf{u}$ is

$$\mathbf{u} = c \sum_{n=1}^{\infty} \sum_{m=0}^{n} (s_n^{mc} \mathbf{S}_n^{mc} + s_n^{ms} \mathbf{S}_n^{ms} + t_n^{mc} \mathbf{T}_n^{mc} + t_n^{ms} \mathbf{T}_n^{ms}) \tag{8}$$

where the s_n^m and t_n^m coefficients are in rad yr^{-1}.

For the rest of the article, we introduce the following more compact notations:

$$b_\alpha \equiv b_{n_\alpha}^{m_\alpha i_\alpha}, \qquad s_\beta \equiv s_{n_\beta}^{m_\beta i_\beta}, \qquad t_\beta \equiv t_{n_\beta}^{m_\beta i_\beta}, \qquad \dot{b}_\gamma \equiv \dot{b}_{n_\gamma}^{m_\gamma i_\gamma},$$

i.e.

$$\alpha \equiv (n_\alpha, m_\alpha, i_\alpha), \qquad \beta \equiv (n_\beta, m_\beta, i_\beta), \qquad \gamma \equiv (n_\gamma, m_\gamma, i_\gamma),$$

and

$$i_\alpha, i_\beta, i_\gamma \in \{c, s\}.$$

2.2 Reduction of the induction equation to a matrix equation

Following the lines of Roberts & Scott (1965), we use expansions (5), (6) and (8) and write equation (3) as a matrix equation:

$$\dot{b}_\gamma = n_\gamma \sum_\beta \left(s_\beta \sum_\alpha (n_\alpha + 1) b_\alpha\, d(\beta, \alpha, \gamma) + t_\beta \sum_\alpha (n_\alpha + 1) b_\alpha\, d'(\beta, \alpha, \gamma)\right) \tag{9a}$$

with the following rules applying:

$$1 \le n_\gamma \le \infty, \qquad 0 \le m_\gamma \le n_\gamma, \qquad i_\gamma \in \{c, s\} \quad \text{and} \quad \sum_\beta \equiv \sum_{n_\beta=1}^{\infty} \sum_{m_\beta=0}^{n_\beta} \sum_{i_\beta \in \{c,s\}}, \qquad \sum_\alpha \equiv \sum_{n_\alpha=1}^{\infty} \sum_{m_\alpha=0}^{n_\alpha} \sum_{i_\alpha \in \{c,s\}}. \tag{10a}$$

The coefficients $d(\beta, \alpha, \gamma)$ and $d'(\beta, \alpha, \gamma)$ are defined by

$$d(\beta, \alpha, \gamma) = \frac{1}{\|\mathbf{T}_\gamma\|^2} J(\beta, \alpha, \gamma), \qquad d'(\beta, \alpha, \gamma) = \frac{1}{\|\mathbf{T}_\gamma\|^2} J'(\beta, \alpha, \gamma), \tag{11}$$

where the coefficients $J(\beta, \alpha, \gamma)$ and $J'(\beta, \alpha, \gamma)$ are

$$J(\beta, \alpha, \gamma) = -c \iint_S Y_\gamma \nabla_H (Y_\alpha \mathbf{S}_\beta)\, dS, \qquad J'(\beta, \alpha, \gamma) = -c \iint_S Y_\gamma \nabla_H (Y_\alpha \mathbf{T}_\beta)\, dS. \tag{12}$$

Equation (9a) can be written in the more compact form

$$\dot{b}_\gamma = \sum_\beta s_\beta M(\gamma, \beta) + \sum_\beta t_\beta M'(\gamma, \beta) \tag{13}$$

or even

$$(\dot{b}) = (M, M')\binom{s}{t} \tag{14}$$

with

$$M(\gamma, \beta) = n_\gamma \sum_\alpha (n_\alpha + 1) b_\alpha d(\beta, \alpha, \gamma), \qquad M'(\gamma, \beta) = n_\gamma \sum_\alpha (n_\alpha + 1) b_\alpha d'(\beta, \alpha, \gamma),$$

and $(\dot{b})$ being a vector the coefficients of which are the $\dot{b}_\gamma$.

As long as all summations are taken up to infinite degrees following the prescribed rules (10a), equations (9a), (13) and (14) are all equivalent to our starting equation (3). But infinite degrees also means an infinite size for the matrix (M, M') which is impossible to handle for practical computation of the flow $(s\ \ t)$ by inversion of equation (14).

2.3 Truncation of the expansions

For this reason and for the even more trivial reason that the coefficients b in expansion (5) and $\dot{b}$ in expansion (6) are known only for low degrees, we are bound to introduce truncations in the expansions used to derive (9a), swapping the summation rules (10a) for the following:

$$1 \le n_\gamma \le N_\gamma, \qquad 0 \le m_\gamma \le n_\gamma, \qquad i_\gamma \in \{c, s\}, \quad \text{and} \quad \sum_\beta = \sum_{n_\beta=1}^{N_\beta} \sum_{m_\beta=0}^{n_\beta} \sum_{i_\beta \in \{c,s\}}, \qquad \sum_\alpha \equiv \sum_{n_\alpha=1}^{N_\alpha} \sum_{m_\alpha=0}^{n_\alpha} \sum_{i_\alpha \in \{c,s\}}, \tag{10b}$$

where N_α (resp. N_γ) is the maximum degree of the MF (resp. SV) that can be used and N_β is the arbitrary maximum degree we choose for the flow we want to calculate from equation (9a).

Swapping rules (10a) for rules (10b) is equivalent to make the very severe assumption that no high-degree terms of either the flow $(n_\beta > N_\beta)$ or the MF $(n_\alpha > N_\alpha)$ interfere in the making of the low degrees of the SV. Doing so, it is quite clear that (9a) is no longer equivalent to equation (3). We are precisely going to explore what kind of error these unavoidable truncations induce.

3 BEHAVIOUR OF THE INTERACTION INTEGRALS $J(\beta, \alpha, \gamma)$ and $J'(\beta, \alpha, \gamma)$

3.1 Selection rules

We define the interaction integrals as the $J(\beta, \alpha, \gamma)$ and $J'(\beta, \alpha, \gamma)$ of equation (12), and the selection rules as conditions that are necessary in order for the interaction integrals to be $\neq 0$. Let us first consider $J(\beta, \alpha, \gamma)$.

Operating the integration over ϕ leads to a first set of selection rules:

$$(i_\beta, i_\alpha, i_\gamma) \in \{(c, c, c); (c, s, s); (s, c, s); (s, s, c)\}, \qquad m_\gamma = m_\alpha + m_\beta \quad \text{or} \quad m_\gamma = |m_\alpha - m_\beta|. \tag{15a}$$

When conditions (15a) are fulfilled $J(\beta, \alpha, \gamma)$ may then be related to the Gaunt and Elsasser integrals:

$$G(\beta, \alpha, \gamma) = \int_0^\pi P_\beta P_\gamma P_\alpha \sin\theta \, d\theta, \qquad E(\beta, \alpha, \gamma) = \int_0^\pi P_\beta \left(n_\gamma P_\gamma \frac{dP_\alpha}{d\theta} - n_\alpha P_\alpha \frac{dP_\gamma}{d\theta} \right) d\theta,$$

where P stands for $P(\cos\theta)$.

These integrals obey some selection rules as well (Gaunt 1929; Bullard & Gellman 1954; Scott 1969) which lead to a second set of rules for the $J(\beta, \alpha, \gamma)$:

$$n_\alpha + n_\beta + n_\gamma \text{ is even}, \qquad (n_\alpha, n_\beta, n_\gamma) \text{ satisfies the condition } |n_\alpha - n_\beta| \le n_\gamma \le n_\alpha + n_\beta. \tag{15b}$$

Considering now the $J'(\beta, \alpha, \gamma)$, it can be shown they obey the following similar rules:

$$(i_\beta, i_\alpha, i_\gamma) \in \{(c, c, s); (s, c, c); (c, s, c); (s, s, s)\}, \qquad m_\gamma = m_\alpha + m_\beta \quad \text{or} \quad m_\gamma = |m_\alpha - m_\beta|, \qquad n_\alpha + n_\beta + n_\gamma \text{ is odd},$$
$$(n_\alpha, n_\beta, n_\gamma) \text{ satisfies the condition } |n_\alpha - n_\beta| \le n_\gamma \le n_\alpha + n_\beta. \tag{16}$$

3.2 Asymptotic behaviour of $J(\beta, \alpha, \gamma)$ and $J'(\beta, \alpha, \gamma)$ for large degrees of the MF and of the flow

$\gamma = (n_\gamma, m_\gamma, i_\gamma)$ is fixed throughout this section and all the n_α and n_β we will consider will be large compared to n_γ (hence to 1).

All couples (α, β) allowed by selection rules (15) satisfy

$$n_\alpha - n_\beta \le n_\gamma \le n_\alpha + n_\beta$$

and therefore

$$n_\alpha \sim n_\beta. \tag{17}$$

In other words, a large degree of the field requires a large degree of the flow in order that $J(\beta, \alpha, \gamma)$ is $\neq 0$. A similar conclusion is reached for $J'(\beta, \alpha, \gamma)$.

For this reason, we expect to be able to find two functions $f(\gamma, n_\alpha)$ and $f'(\gamma, n_\alpha)$ such that the asymptotic values of $J(\beta, \alpha, \gamma)$ and $J'(\beta, \alpha, \gamma)$ when n_α and n_β are large compared to n_γ, satisfy

$$|J(\beta, \alpha, \gamma)| \leq f(\gamma, n_\alpha) \quad \text{and} \quad |J'(\beta, \alpha, \gamma)| \leq f'(\gamma, n_\alpha). \tag{18}$$

Considering for instance $J'(\beta, \alpha, \gamma)$, we have

$$c\nabla_H(Y_\alpha \mathbf{T}_\beta) = \mathbf{S}_\alpha \cdot \mathbf{T}_\beta$$

so that (12) becomes

$$J'(\beta, \alpha, \gamma) = -\iint_S Y_\gamma(\mathbf{S}_\alpha \cdot \mathbf{T}_\beta)\, dS$$

as $n_\gamma \ll n_\alpha, n_\beta$, Y_γ is much smoother a function than Y_α and Y_β. Since we know that $|Y_\gamma(\theta, \phi)|$ is bounded by a constant value D_γ, we may find a reasonable bound using the following inequalities:

$$|J'(\beta, \alpha, \gamma)| \leq D_\gamma \iint_S |\mathbf{S}_\alpha \cdot \mathbf{T}_\beta|\, dS \leq D_\gamma\, \|\mathbf{S}_\alpha\|\, \|\mathbf{T}_\beta\|$$

leading to

$$|J'(\beta, \alpha, \gamma)| \leq 2\pi D_\gamma\left(n_\alpha + \frac{n_\gamma + 1}{2}\right) \tag{19b}$$

whatever the values for (β, α, γ) allowed by the selection rules (16).

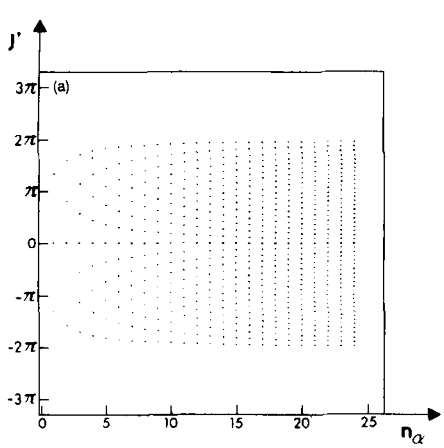
A similar calculation leads to the less simple result

$$|J(\beta, \alpha, \gamma)| \leq 4\pi D_\gamma(n_\alpha + \tfrac{7}{4} n_\gamma) \tag{19a}$$

whatever the values for (β, α, γ) allowed by the selection rules (15) as long as they also satisfy the requirement $n_\alpha \geq n_\gamma$. Equations (19a) and (19b) are indeed of the form we expected [recall (18)].

To see how rough inequalities (19a) and (19b) may be, we computed numerically some of the values of $J(\beta, \alpha, \gamma)$ and $J'(\beta, \alpha, \gamma)$. The results are shown in Figs 1–4. Each figure corresponds to a given γ [i.e. $(n_\gamma, m_\gamma, i_\gamma)$] of the SV and we plotted all $J(\beta, \alpha, \gamma)$ [resp. $J'(\beta, \alpha, \gamma)$] allowed by selection rules (15) [resp. (16)] for each degree n_α (up to degree $25 - n_\gamma$) of the MF.

There are many **J** and **J**′ for each n_α, but we know they all satisfy inequalities (19a) and (19b) for large n_α. This is of course what we see on Figs 1–4. But the numerical results also strongly suggest there might be a bound for the **J** and **J**′ that



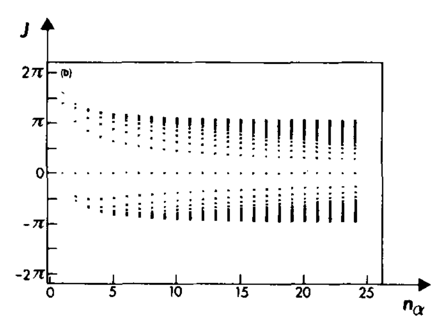


Figure 1. See Section 3.2. As a function of n_α, the degree of the Main Field. (a) J′ interaction integrals for $n_\gamma = 1$, $m_\gamma = 0$, i_γ = cosine. (b) J interaction integrals for $n_\gamma = 1$, $m_\gamma = 0$, i_γ = cosine.

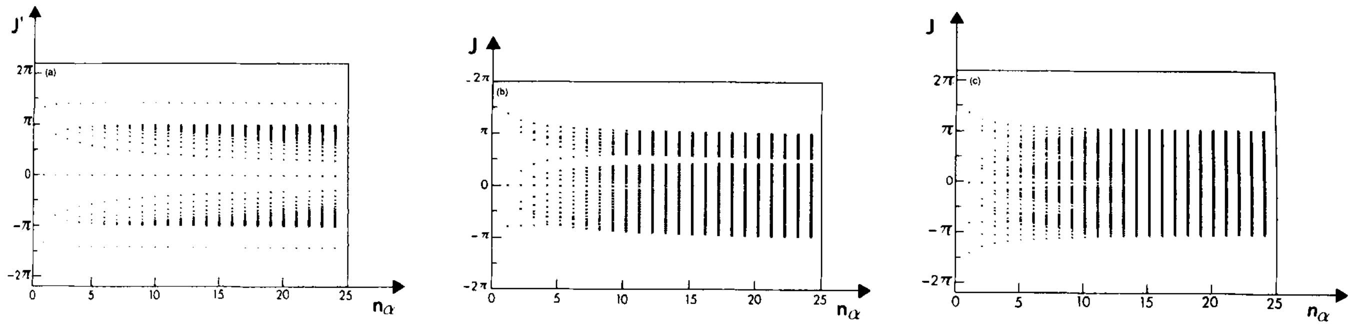


Figure 2. See Section 3.2. As a function of n_α, the degree of the Main Field. (a) J′ interaction integrals for $n_\gamma = 1$, $m_\gamma = 1$, $i_\gamma =$ cosine or sine. (b) J interaction integrals for $n_\gamma = 1$, $m_\gamma = 1$, $i_\gamma =$ cosine. (c) J interaction integrals for $n_\gamma = 1$, $m_\gamma = 1$, $i_\gamma =$ sine.

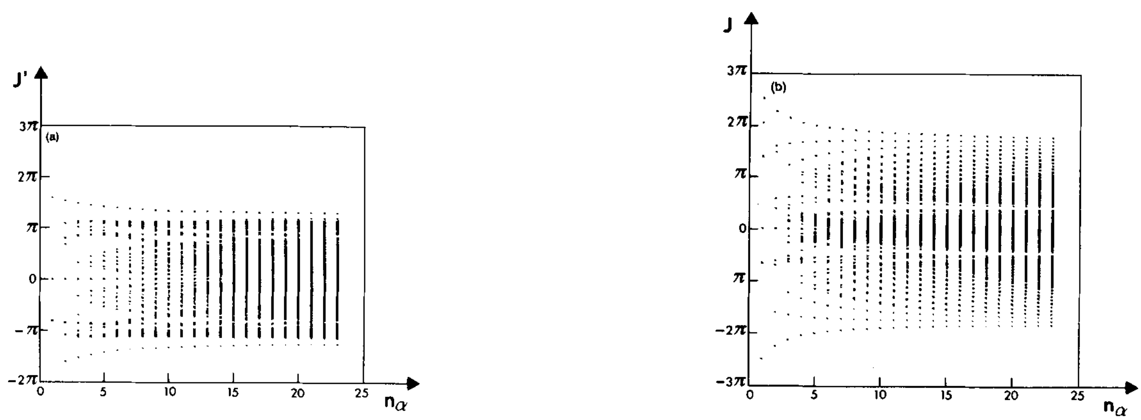


Figure 3. See Section 3.2. As a function of n_α, the degree of the Main Field. (a) J′ interaction integrals for $n_\gamma = 2$, $m_\gamma = 2$, $i_\gamma =$ cosine. (b) J interaction integrals for $n_\gamma = 2$, $m_\gamma = 2$, $i_\gamma =$ sine.

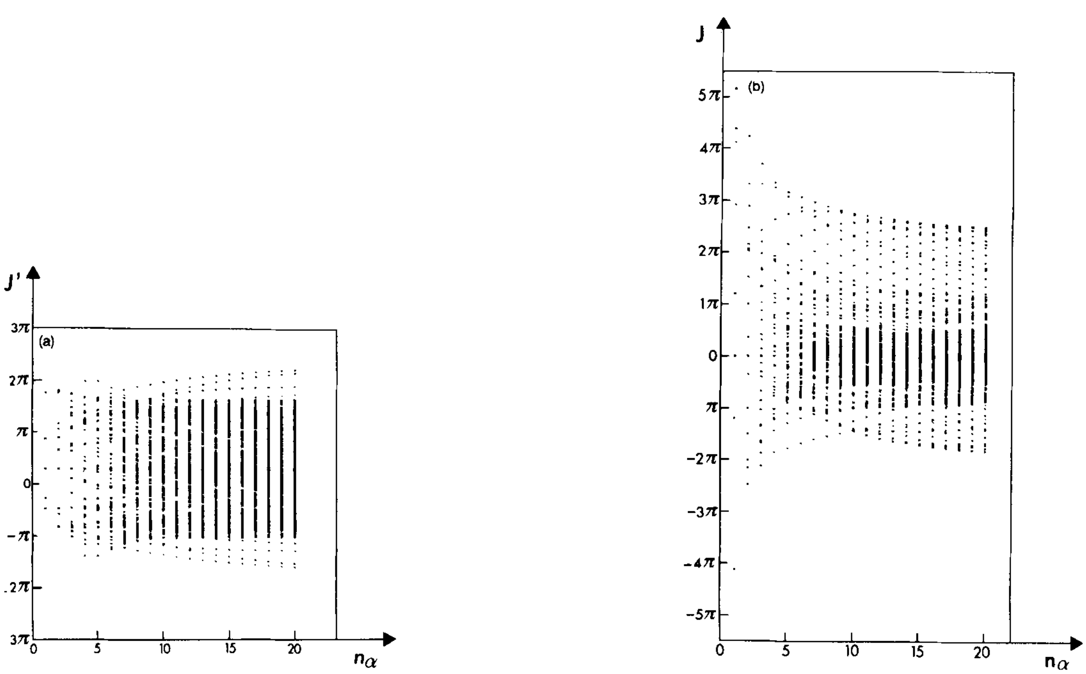


Figure 4. See Section 3.2. As a function of n_α, the degree of the Main Field. (a) J′ interaction integrals for $n_\gamma = 5$, $m_\gamma = 4$, $i_\gamma =$ sine. (b) J interaction integrals for $n_\gamma = 5$, $m_\gamma = 4$, $i_\gamma =$ cosine.

does not depend on n_α, in other words that we have

$$|J(\beta, \alpha, \gamma)| \le \pi K(\gamma), \tag{20a}$$

$$|J'(\beta, \alpha, \gamma)| \le \pi K'(\gamma). \tag{20b}$$

These inequalities are not as easy to derive as inequalities (19a) and (19b). We have been able to provide a proof of (20a), using generalized spherical harmonics. But as this proof is somehow laborious, we won't give it here. In fact we will only need an estimate of the root mean square (RMS) value $\langle J(n_\alpha, \gamma)\rangle$ of quantities $J(\beta, \alpha, \gamma)$ and $J'(\beta, \alpha, \gamma)$ for a given n_α and γ, as defined by

$$\langle J(n_\alpha, \gamma)\rangle = \left(\frac{\left\{\sum_{m_\alpha, i_\alpha, \beta} [J(\beta, \alpha, \gamma)]^2 + \sum_{m_\alpha, i_\alpha, \beta} [J'(\beta, \alpha, \gamma)]^2\right\}}{N(n_\alpha, \gamma) + N'(n_\alpha, \gamma)}\right)^{1/2} \tag{21}$$

where the first summation is performed over all the $N(n_\alpha, \gamma)$ possible values for $(m_\alpha, i_\alpha, \beta)$ from (15) and the second over all the $N'(n_\alpha, \gamma)$ possible values for $(m_\alpha, i_\alpha, \beta)$ from (16).

The result suggested by (20) is that $\langle J(n_\alpha, \gamma)\rangle$ remains finite when n_α becomes large. Indeed (20) implies

$$\langle J(n_\alpha, \gamma)\rangle \le \pi \sqrt{\frac{[N(n_\alpha, \gamma)K^2(\gamma) + N'(n_\alpha, \gamma)K'^2(\gamma)]}{[N(n_\alpha, \gamma) + N'(n_\alpha, \gamma)]}}.$$

In fact we can prove (see Appendix A) that one can exactly describe the behaviour of the $\langle J(n_\alpha, \gamma)\rangle$ when n_α becomes large with the following formulae:

$$\langle J(n_\alpha, \gamma)\rangle \sim \frac{\sqrt{n_\gamma(n_\gamma+1)}}{n_\gamma + \frac{1}{2}} \times \begin{cases} \pi & \text{if } m_\gamma = 0, \\ \frac{1}{\sqrt{2}}\pi & \text{if } m_\gamma \ne 0. \end{cases} \tag{22a}$$

This result is in perfect agreement with the numerical results of Fig. 5. As can be easily checked, the factor $[n_\gamma(n_\gamma+1)]^{1/2}(n_\gamma + 1/2)^{-1}$ is very close to 1 (it lies between 0.94 and 1), so that from a practical point of view, rather than (22a), we will assume

$$\langle J(n_\alpha, \gamma)\rangle \sim \begin{cases} \pi & \text{if } m_\gamma = 0, \\ \frac{1}{\sqrt{2}}\pi & \text{if } m_\gamma \ne 0. \end{cases} \tag{22b}$$

(22b) will soon prove useful (Section 4.2 and following).

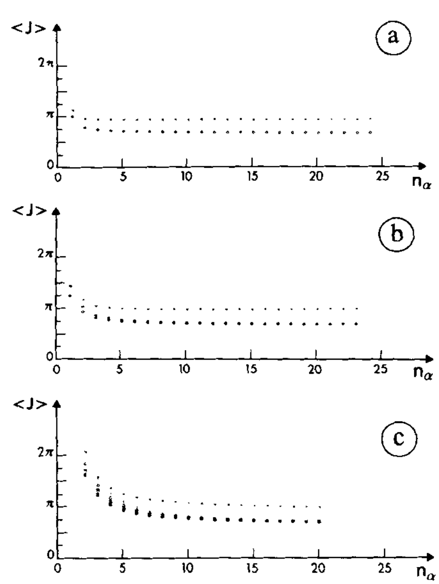


Figure 5. See Section 3.2, formula (21). RMS values $\langle J(n_\alpha, \gamma)\rangle$ of the J and J′ interaction integrals when (n_α, γ) is given, as a function of n_α, the degree of the Main Field, for: (a) SV degree $n_\gamma = 1$; (b) SV degree $n_\gamma = 2$; and (c) SV degree $n_\gamma = 5$. In all three cases, those values lying well above the others correspond to the special case $m_\gamma = 0$.

4 ESTIMATION OF THE ERRORS ON THE COMPUTED SV COEFFICIENTS DUE TO THE TRUNCATION OF THE EXPANSIONS OF THE FLOW AND OF THE MF IN THE INDUCTION EQUATION

4.1 Estimations of the field and of the flow at large degrees

The energy of the magnetic field at the CMB (within a multiplying constant) is (Lowes 1966)

$$W = \frac{1}{4\pi c^2} \iint_{\text{Core}} (\mathbf{B}\cdot\mathbf{B})\, dS = \sum_{n_\alpha=1}^{\infty} W(n_\alpha)$$

with

$$W(n_\alpha) = (n_\alpha+1) \sum_{m_\alpha=0}^{n_\alpha} [(b_{n_\alpha}^{m_\alpha c})^2 + (b_{n_\alpha}^{m_\alpha s})^2].$$

The spectrum of the main field [i.e. the collection of the $W(n_\alpha)$] is now well known. It features a slope change at degree 13 indicative of a strong crustal field for degrees greater than 13 (Langel & Estes 1982). We therefore know that it is difficult if ever possible to infer the spectrum of the core-generated MF for degrees greater than 13. On the other hand, we know that for degrees less than 13 the energy of the crustal field is much smaller than that of the MF [for degree 13, the contribution of the crustal field does not exceed 20 per cent of W (Counil, Cohen & Achache 1991)], so that it is possible to assume that this part of the spectrum is of internal origin. Fig. 6(a) shows $W(n_\alpha)$ corresponding to a model of the field for the year 1980 up to degree 13 continued to the core (Cohen & Achache 1990). (In fact any other model for the same period would give the same figure.) One infers from this drawing that $W(n_\alpha)$ can be considered as obeying an exponential law for large enough n_α:

$$W(n_\alpha) \sim W_0 e^{-kn_\alpha} \quad \text{with} \quad k \approx 0.14, \quad W_0 \approx 2\times 10^{10}\,(\text{nT})^2. \tag{23}$$

It is a reasonable assumption that equation (23) describing the behaviour of the core field spectrum should remain valid for degrees n_α larger than 13.

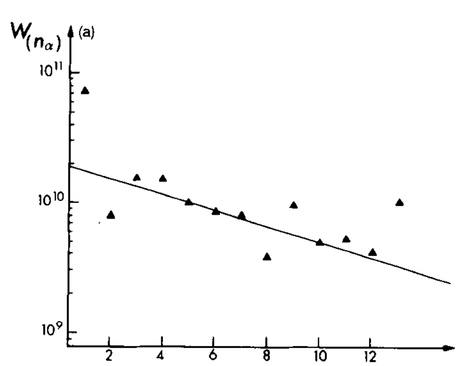
For a given n_α, we now define $b(n_\alpha)$ as a random centred variable, its RMS value $\langle b(n_\alpha)\rangle$ being such that

$$W_0 e^{-kn_\alpha} = (n_\alpha+1)(2n_\alpha+1)\langle b(n_\alpha)\rangle^2$$

so we have (for large n_α):

$$\langle b(n_\alpha)\rangle = \sqrt{\frac{W_0}{(n_\alpha+1)(2n_\alpha+1)}}\, e^{-(k/2)n_\alpha} \sim \sqrt{\frac{W_0}{2}}\, \frac{e^{-(k/2)n_\alpha}}{n_\alpha}. \tag{24}$$

We argue that for a given n_α, the b_α MF coefficients may be viewed as independently drawn lots of the random variable $b(n_\alpha)$. We can indeed calculate the RMS value $\langle b(\alpha)\rangle$ and the mean value $\bar{b}_\alpha$ of the b_α coefficients for each degree n_α of the MF model and compare them respectively to $\langle b(n_\alpha)\rangle$ from (24) and to the mathematical expectation of $b(n_\alpha)$, $E[b(n_\alpha)]=0$ (see Fig. 6b). The agreement is fairly good, the discrepancy being easily explained by the finite number $(2n_\alpha+1)$ of b_α coefficients: in the case of the mean value, for instance, we know that $\bar{b}_\alpha$ is allowed to fluctuate with an RMS amplitude of $\langle b(n_\alpha)\rangle/\sqrt{2n_\alpha+1}$ about the value $E[b(n_\alpha)]=0$. Too quick a glance at Fig. 6(b) might give the uncomfortable feeling that the



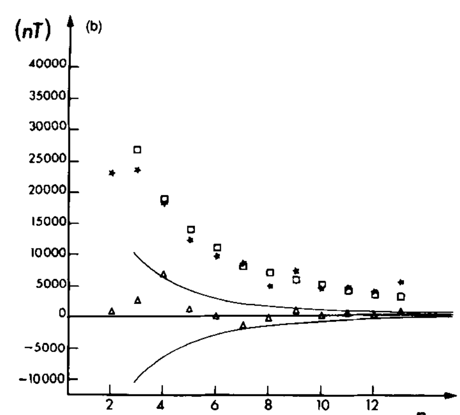


Figure 6. See Sections 4.1 and 4.2. (a) Spectrum of the magnetic field at the CMB [$W(n_\alpha)$ is in $(\text{nT})^2$]. Also shown (solid line), the values given by formula (23). (b) As a function of n_α, the degree of the Main Field (degree 1 not shown): the RMS value $\langle b(n_\alpha)\rangle$ for the model from Cohen & Achache (1990) (stars); the mean value $\bar{b}_\alpha$ (open triangles); the RMS value $\langle b(n_\alpha)\rangle$ given by formula (24) (open squares); the fluctuation 'amplitude' for $\bar{b}_\alpha$ (solid lines).

$\bar{b}_\alpha$ do not fluctuate as much as expected. Let us make a very simple reasoning in order to see if this distribution is really too unlikely; the $\bar{b}_\alpha$ seem to be too often (10 times out of 12) within the RMS amplitude $\langle b(n_\alpha)\rangle/\sqrt{2n_\alpha+1}$ and too seldom above this amplitude. If we assume that we deal with Gaussian random variables, what would most likely happen is that eight out of the 12 $\bar{b}_\alpha$ would be within this amplitude and four above. The probabilities corresponding to these two cases are respectively $C_{12}^2 \times (0.69)^{10} \times (0.31)^2 = 0.155$ and $C_{12}^4 \times (0.69)^8 \times (0.31)^4 = 0.235$, which proves that the observed case is nearly as likely to happen as the most likely case: we may assume the b_α coefficients are independent drawing lots of the random variable $b(n_\alpha)$.

There is still another difficulty linked to this statement: since the values of the b_α coefficients rely on the choice of the longitude's origin, would the statement still be true if this origin was to be shifted? This important point is discussed in Appendix B. It is shown that (hopefully) the statement does not rely on the longitude's origin.

We can define the energy of the flow in the same way as we did for the MF:

$$E = \frac{1}{4\pi c^4} \iint_{\text{Core}} (\mathbf{u}\cdot\mathbf{u})\, dS = \sum_{n_\beta=1}^{\infty} E_S(n_\beta) + \sum_{n_\beta=1}^{\infty} E_T(n_\beta).$$

Using expansion (8), we can write

$$E_S(n_\beta) = \frac{(n_\beta+1)n_\beta}{2n_\beta+1} \sum_{m_\beta=0}^{n_\beta} [(s_{n_\beta}^{m_\beta c})^2 + (s_{n_\beta}^{m_\beta s})^2], \qquad E_T(n_\beta) = \frac{(n_\beta+1)n_\beta}{2n_\beta+1} \sum_{m_\beta=0}^{n_\beta} [(t_{n_\beta}^{m_\beta c})^2 + (t_{n_\beta}^{m_\beta s})^2]. \tag{25}$$

We mentioned in the introduction that, in order to obtain a practical uniqueness, the flow at the CMB must be supposed to be large scale. This implies at least

$$E_S = \sum_{n_\beta=1}^{\infty} E_S(n_\beta) < \infty \quad \text{and} \quad E_T = \sum_{n_\beta=1}^{\infty} E_T(n_\beta) < \infty. \tag{26}$$

In order that (26) holds we must suppose that for n_β large enough,

$$E_S(n_\beta) \sim E_T(n_\beta) \sim E(n_\beta) = \frac{E_0}{(n_\beta)^l} \quad \text{with} \quad l > 1, \tag{27}$$

where E_0 is a constant we will estimate in Section 4.3. Here again we introduce a statistical formalism to describe a typical flow $(s\ t)$. For a given n_β, we will define $u(n_\beta)$ as a random variable such that the s_β and t_β may be viewed as independently drawn lots of this random variable. Not much is known *a priori* about $u(n_\beta)$ but we expect and we will assume that

$$E[u(n_\beta)] = 0.$$

(It is easy to make sure *a posteriori* that this is reasonable using, for instance, the 'typical' flow derived in Section 5.1.) And, on the basis of (25) and (27), we obtain for large n_β:

$$\langle u(n_\beta)\rangle \sim \sqrt{E_0}\frac{1}{n_\beta^{1+(l/2)}}. \tag{28}$$

Relations (28) are used as '*a priori*' information in the inversion schemes (e.g. Backus 1988). It is worth noting that this statistical assumption raises the same question as the statistical assumption we used for the MF: is it independent of the choice of the longitude's origin? The arguments developed in Appendix B in the case of the MF can easily be extended to the case of the flow so that the answer is the same. None of the statistical assumptions we make for either the MF or the flow relies on this choice and of course, the same thing will be true for the conclusions we will reach in this paper.

4.2 Contribution of large degree terms of the field and of the flow to the induction equation

Returning to equation (9a) we can write it in the form

$$\dot{b}_\gamma = \sum_{n_\alpha=1}^{\infty} B(\gamma, n_\alpha) \tag{9b}$$

with

$$B(\gamma, n_\alpha) = n_\gamma(n_\alpha+1) \sum_{m_\alpha=0}^{n_\alpha} \sum_{i_\alpha \in \{c,s\}} \sum_{\beta} b_\alpha[s_\beta d(\beta,\alpha,\gamma) + t_\beta d'(\beta,\alpha,\gamma)]. \tag{29}$$

$B(\gamma, n_\alpha)$ is the contribution of the components of the MF of degree n_α to the SV of index γ when acted by the flow $(s\ t)$.

Equation (29) involves *a priori* an infinite sum on β. But as already mentioned for equation (21), the selection rules (15) and (16) which apply to the $J(\beta,\alpha,\gamma)$ and $J'(\beta,\alpha,\gamma)$ and therefore to the $d(\beta,\alpha,\gamma)$ and $d'(\beta,\alpha,\gamma)$ make it finite. Indeed γ being given, if we choose a large n_α, it follows from (15) [resp. (16)] that the number $N(n_\alpha,\gamma)$ [resp. $N'(n_\alpha,\gamma)$] of non-zero

$d(\beta, \alpha, \gamma)$ [resp. $d'(\beta, \alpha, \gamma)$] involved in (29)—please note that these are the same $N(n_\alpha, \gamma)$ and $N'(n_\alpha, \gamma)$ as those involved in (21)—is about $2n_\alpha(n_\gamma+1)$ (resp. $2n_\alpha n_\gamma$) if $m_\gamma=0$, and $4n_\alpha(n_\gamma+1)$ (resp. $4n_\alpha n_\gamma$) otherwise. We want to estimate the magnitude of $B(\gamma, n_\alpha)$ for large n_α.

Considered from the statistical point of view we adopted for the description of the MF and of the flow, (29) is the summation of products of independent draws of several random variables [for instance $b_{n_\alpha}^{m_\alpha c}$, $b_{n_\alpha}^{m_\alpha s}$ and $b_{n_\alpha}^{m'_\alpha c}$ are three independent draws of the same random variable $b(n_\alpha)$, and $b(n_\alpha)$, $b(n'_\alpha)$ and $u(n_\beta)$ are independent random variables; the assumption that $u(n_\beta)$ and $b(n_\alpha)$ are independent variables could be discussed. But to know whether it holds exactly is not important for the order of magnitude calculation we want to make in this paper: the interactions between the flow and the field are so complex that it would require very sophisticated correlations between $u(n_\beta)$ and $b(n_\alpha)$ for these interactions to act in a coherent manner and significantly change our results]. Then, recalling that $E[b(n_\alpha)]=0$ and $E[u(n_\beta)]=0$, we have from (29):

$$E[B(\gamma, n_\alpha)]=0 \tag{30}$$

and, for the RMS value:

$$\langle B(\gamma, n_\alpha)\rangle = n_\gamma(n_\alpha+1)\langle b(n_\alpha)\rangle \left\{ \sum_{m_\alpha=0}^{n_\alpha} \sum_{i_\alpha \in \{c,s\}} \sum_{\beta} \langle u(n_\beta)\rangle^2 [d^2(\beta, \alpha, \gamma) + d'^2(\beta, \alpha, \gamma)] \right\}^{1/2}. \tag{31a}$$

From (17) and (28) we know that for n_α large compared to n_γ

$$\langle u(n_\beta)\rangle \sim \langle u(n_\alpha)\rangle.$$

This allows us to write, also using (11) and (22b),

$$\langle B(\gamma, n_\alpha)\rangle \sim \frac{n_\gamma(n_\alpha+1)}{\|T_\gamma\|^2} \langle b(n_\alpha)\rangle \langle u(n_\alpha)\rangle \langle J(n_\alpha, \gamma)\rangle [N(n_\alpha, \gamma) + N'(n_\alpha, \gamma)]^{1/2}. \tag{31b}$$

The previous considerations provide a statistical estimate for the magnitude of $B(\gamma, n_\alpha)$. For n_α large compared to n_γ (31b) becomes [recall (7), (24), (28) and (22a)]

$$\langle B(\gamma, n_\alpha)\rangle \sim g(\gamma) B_0 B(n_\sigma) \tag{31c}$$

with

$$g(\gamma) = \frac{(2n_\gamma+1)^{3/2}}{4(n_\gamma+1)}, \qquad B_0 = \sqrt{E_0 W_0}, \qquad B(n_\alpha) = \frac{e^{-(k/2)n_\alpha}}{n_\alpha^{1/2(1+l)}}$$

where $g(\gamma)$ describes the dependence on the degree of the created SV, $B(n_\alpha)$ the dependence on the degree of the MF, and B_0 has the dimension of a SV.

Let us turn now to the core of the problem. Define what we ought to call the 'rest of the SV', i.e. the summation of all the $B(\gamma, n_\alpha)$ that have been neglected because of the truncation (Section 2):

$$R(\gamma, N_\alpha) = \sum_{n_\alpha=N_\alpha+1}^{\infty} B(\gamma, n_\alpha).$$

The 'rest of the SV' will be our means to measure the error on the computed SV coefficients due to the truncation of the expansions of the flow and the MF in the induction equation. To evaluate $R(\gamma, N_\alpha)$, we will go on using the statistical formalism already introduced. The assumptions made about the statistical independence of the $b(n_\alpha)$ and the $u(n_\beta)$ allow us to treat the different $B(\gamma, n_\alpha)$ as also statistically independent. From (30) we then have

$$E[R(\gamma, N_\alpha)]=0$$

and

$$\langle R(\gamma, N_\alpha)\rangle \sim \left(\sum_{n_\alpha=N_\alpha+1}^{\infty} \langle B(\gamma, n_\alpha)\rangle^2 \right)^{1/2}$$

and using (31c), this can be written

$$\langle R(\gamma, N_\alpha)\rangle \sim g(\gamma) B_0 \left(\sum_{n_\alpha=N_\alpha+1}^{\infty} B(n_\alpha)^2 \right)^{1/2}. \tag{32}$$

Thus equation (32) provides a statistical answer to our question.

The above result has been derived assuming among other things that $E[b(n_\alpha)]$ and $E[u(n_\beta)]$ are zero. Although we showed this is compatible with the data, it might not exactly be the case. As one can see on Fig. 6(b), stating

$$E[b(n_\alpha)] = \epsilon_b(n_\alpha)\langle b(n_\alpha)\rangle \tag{33}$$

where $\epsilon_b(n_\alpha)$ has a constant ϵ_b absolute value and a random sign (function of n_α) and assuming that ϵ_b is not larger than say 0.2, is as plausible a statement as $E[b(n_\alpha)]=0$. In the same way, rather than assuming $E([u(n_\beta)]=0$, we could as well assume

$$E[u(n_\beta)]=\epsilon_u(n_\beta)\langle u(n_\beta)\rangle \tag{34}$$

where $\epsilon_u(n_\beta)$ also has a constant ϵ_u absolute value and a random sign (now function of n_β); ϵ_μ can be supposed smaller than say 0.3 from the 'typical' flow derived in Section 5.1. Fortunately such corrections will not affect the result (32), as is shown in Appendix C.

We now have a value at our disposal to evaluate the SV error $R(\gamma, N_\alpha)$ due to the truncation of the induction equation.

4.3 Numerical results

In this section we will derive some numerical values for the $\langle R(\gamma, N_\alpha)\rangle$. To do so, we need some values for the not yet known parameters E_0 and l. Recalling (27), we note that the value of E_0 depends on the value chosen for l, in other words $E_0=E_0(l)$. On the basis of a number of published models of the flow at the CMB (Le Mouël *et al.* 1985; Voorhies 1986; Gire, Le Mouël & Madden 1986; Whaler & Clarke 1988; Bloxham 1989; Bloxham *et al.* 1989; Gire & Le Mouël 1990), a fair way of defining the value of $E_0(l)$ is to make sure that

$$E(n_\beta=10)=\frac{E_0(l)}{10^l}\sim 10^{-6}\,\mathrm{rad}^2\,\mathrm{yr}^{-2}.$$

It remains to choose the value of the rather arbitrary parameter l. A standard value for l is 2 (Backus 1988; Gire & Le Mouël 1990), but we also made the calculation for $l=3$ and $l=1.1$ in order to explore the effect of the choice of l.

Eventually we need the truncation degree N_α of the MF. This value is directly given by the observation: $N_\alpha=13$, since 13 is the maximum degree (because of the crustal field) we can reasonably resolve for the MF. We therefore present three sets of results, (l, N_α) being: (1.1, 13), (2, 13) and (3, 13). Table 1 gives the numerical values of $g(\gamma)$, showing its evolution with the degree n_γ of the SV. Table 2 illustrates the fact that the value of our estimation for $R(\gamma, N_\alpha)$ is fairly independent of the value we choose for l. For this reason we will keep 2 as a reasonable value for l. Table 3 then gives our final estimate for $R(\gamma, N_\alpha)$ with $N_\alpha=13$.

Because the calculation that led to formula (32) is based on the use of asymptotic expressions, we would like to confront our result with some kind of 'real' data. Of course we do not have the data that would enable us to confront the true $R(\gamma, N_\alpha)$ with our estimation $\langle R(\gamma, N_\alpha)\rangle$. But it is nevertheless possible to assess the validity of our approach. Using the typical model for the flow (Section 5.1) and a model of the main field (e.g. Cohen & Achache 1990), we can easily calculate the contribution to the SV of the field components of degrees 11 to 13, i.e. the 'real' $R(\gamma, 10)-R(\gamma, 13)$. Using the same statistical approach as the one used to derive $\langle R(\gamma, N_\alpha)\rangle$, we can make a statistical estimation of this difference, $\langle R(\gamma, 10)-R(\gamma, 13)\rangle$ (see Table 4). Fig. 7 shows both the 'real' RMS value of $R(\gamma, 10)-R(\gamma, 13)$ and the RMS statistical estimation $\langle R(\gamma, 10)-R(\gamma, 13)\rangle$. Our statistical estimation is in fairly good agreement with the 'real' case. We therefore consider formula (32) as a realistic estimation for $R(\gamma, N_\alpha)$.

Table 1. $g(\gamma)$ (equation 31) as a function of the SV degree n_γ (see Section 4.3).

n_γ	$g(\gamma)$
1	0.650
2	0.932
3	1.158
4	1.350
5	1.520
6	1.674
7	1.815
8	1.947

Table 2. Dependence of $\langle R(\gamma, 13)\rangle/g(\gamma)$ with l (in nT yr^{-1}) (see Section 4.3).

	$l=1.1$	$l=2$	$l=3$
$\langle R(\gamma,13)\rangle/g(\gamma)$	24.6	19.4	15.2

Table 3. Final estimate for $R(\gamma, N_\alpha)$ with $N_\alpha=13$ (in nT yr^{-1}) (see Section 4.3).

n_γ	$\langle R(\gamma,13)\rangle$
1	12.6
2	18.1
3	22.5
4	26.2
5	29.5
6	32.5
7	35.2
8	37.8

Table 4. Statistical estimation for $R(\gamma, 10)-R(\gamma, 13)$ (in nT yr^{-1}) (see Section 4.3).

n_γ	$\langle R(\gamma,10)-R(\gamma,13)\rangle$
1	17.0
2	24.4
3	30.3
4	35.4
5	39.8
6	43.9
7	47.6
8	51.0

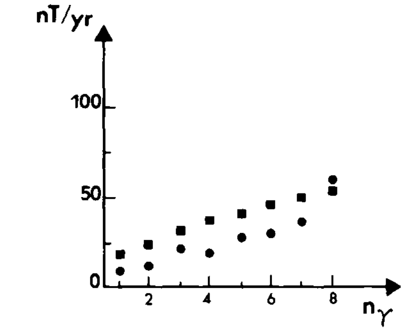


Figure 7. See Section 4.3. As a function of n_γ, the SV degree: the 'real' RMS value of $R(\gamma, 10) - R(\gamma, 13)$ (full circles); the RMS statistical estimation $\langle R(\gamma, 10) - R(\gamma, 13)\rangle$ (full squares).

4.4 Implications for the calculation of the flow at the CMB

We now need to compare $\langle R(\gamma, N_\alpha)\rangle$ to the $\dot{b}_\gamma$ coefficients of the SV. This is the purpose of Fig. 8(a) which shows $\langle R(\gamma, N_\alpha)\rangle$ together with the RMS value of the $\dot{b}_\gamma$ (at the CMB) as a function of n_γ. The most important result is that the truncation errors lie well below the RMS values of the $\dot{b}_\gamma$. This proves that the principle of deriving a flow (within the assumed hypothesis) using an unavoidably truncated expansion of the field has not to be questioned. But it does not mean that the error is negligible and its estimation of no interest. Thanks to this estimate we are indeed able to give some indications on which part of the flow could ultimately be resolved from the magnetic data.

Considering the truncated form of equation (9b) (Section 4.2):

$$\dot{b}_\gamma = \sum_{n_\alpha=1}^{N_\alpha} B(\gamma, n_\alpha) \quad \text{with} \quad N_\alpha = 13, \tag{9c}$$

and recalling that the SV is hardly known for n_γ greater than 8, we know that only the degrees of the flow less than 21 will be constrained by (9c) (i.e. by the data). Of course, not all degrees of the flow will be constrained in the same way: the greater the degree, the poorer the constraint.

Define $B(\gamma, n_\alpha, N_\beta)$, the contribution to $B(\gamma, n_\alpha)$ of the degrees of the flow greater than a given degree N_β:

$$B(\gamma, n_\alpha, N_\beta) = n_\gamma(n_\alpha + 1) \sum_{m_\alpha=0}^{n_\alpha} \sum_{i_\alpha \in \{c,s\}} \sum_{\beta'} b_\alpha[s_\beta d(\beta, \alpha, \gamma) + t_\beta d'(\beta, \alpha, \gamma)]$$

β' meaning β such that $n_\beta > N_\beta$, and

$$R(\gamma, N_\alpha, N_\beta) = \sum_{n_\alpha=1}^{N_\alpha} B(\gamma, n_\alpha, N_\beta)$$

the contribution to $\dot{b}_\gamma$ in (9c) of the flow components with degree greater than N_β.

We can make a statistical estimation of $B(\gamma, n_\alpha, N_\beta)$ and $R(\gamma, N_\alpha, N_\beta)$ in the way we did for $B(\gamma, n_\alpha)$ and $R(\gamma, N_\alpha)$. We will use the same hypothesis for the field and the flow as previously but we won't need any asymptotic estimation for the J and J' (only a finite number of them are involved in the calculation). We obtain for the RMS value of $B(\gamma, n_\alpha, N_\beta)$:

$$\langle B(\gamma, n_\alpha, N_\beta)\rangle = \frac{n_\gamma(n_\alpha + 1)}{\|\mathbf{T}_\gamma\|^2} \langle b(n_\alpha)\rangle \sum_{\beta'} \langle u(n_\beta)\rangle \left\{ \sum_{m_\alpha=0}^{n_\alpha} \sum_{i_\alpha \in \{c,s\}} [J^2(\beta, \alpha, \gamma) + J'^2(\beta, \alpha, \gamma)] \right\}^{1/2}$$

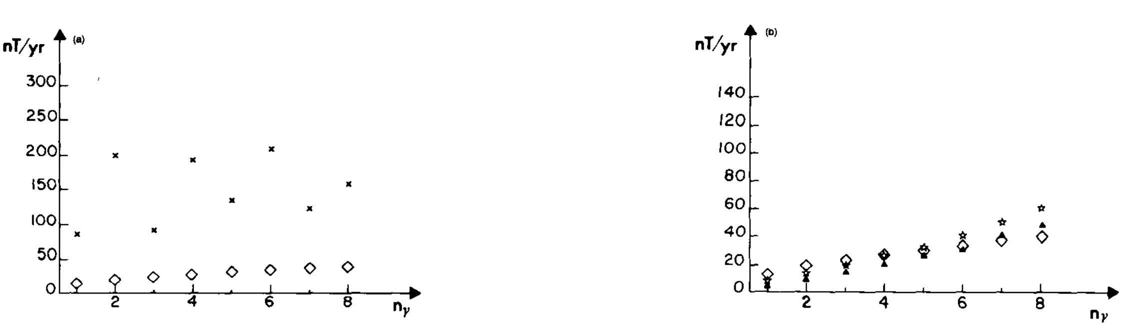


Figure 8. See Section 4.4. As a function of n_γ, the SV degree: (a) the RMS value of the $\dot{b}_\gamma$ at the CMB (crosses): the $\langle R(\gamma, N_\alpha = 13)\rangle$ (open diamonds); (b) the $\langle R(\gamma, N_\alpha = 13)\rangle$ (open diamonds); the RMS value of the $\langle R(\gamma, N_\alpha = 13,\ N_\beta = 13)\rangle$ over the several possible m_γ (the $\langle R(\gamma, N_\alpha, N_\beta)\rangle$ have little dependence on m_γ) (full triangles); same for the $\langle R(\gamma, N_\alpha = 13,\ N_\beta = 12)\rangle$ (open stars).

and for the RMS value of $R(\gamma, N_\alpha, N_\beta)$:

$$\langle R(\gamma, N_\alpha, N_\beta)\rangle = \left(\sum_{n_\alpha=1}^{N_\alpha} \langle B(\gamma, n_\alpha, N_\beta)\rangle^2\right)^{1/2}. \quad (35)$$

Recalling (24) and (28), (35) allows us to calculate $\langle R(\gamma, N_\alpha, N_\beta)\rangle$ for any given N_β.

We now simply argue that if $\langle R(\gamma, N_\alpha, N_\beta)\rangle$ is less than or of the order of the truncation error $\langle R(\gamma, N_\alpha)\rangle$, the magnetic are unable to constrain the components of the flow of degree greater than N_β. We made the calculation for various N_β of interest and Fig. 8(b) shows our results. As can be seen, $\langle R(\gamma, N_\alpha = 13, N_\beta = 13)\rangle$ is always less or about the size of $\langle R(\gamma, N_\alpha = 13)\rangle$ whereas $\langle R(\gamma, N_\alpha = 13, N_\beta = 12)\rangle$ is already quite comparable to or even larger than $\langle R(\gamma, N_\alpha = 13)\rangle$.

The conclusion is that the error due to the truncation of the expansion of the field makes it impossible to derive any information on the degrees of the flow greater than 12. As a pleasant consequence this however proves that we may truncate the expansion of the flow at degree 12 before inverting equation (14).

5 DERIVING THE FLOW AT THE CMB: A SIMPLE EVALUATION OF THE PRESENT-DAY ACCURACY

The previous study was made assuming that the only errors that were allowed to creep in the calculation were those linked to the truncation of the expansions. This gave us a hint of what information could ultimately be extracted from the magnetic data in the case the Gauss coefficients $\dot{b}_\gamma$ would be perfectly known. Unfortunately, such is not the case. We must take this important fact into account to clarify what can be said about the flow with the present-day magnetic data.

5.1 The error on the SV data

It is not an easy task to evaluate the errors that are made on the b_γ in the process of deriving a model of the SV from the observations. But for the following discussion, we will assume that comparing two available models for the year 1980 [i.e. computing the differences $\Delta\dot{b}_\gamma = \dot{b}_\gamma$(USGS80) $- \dot{b}_\gamma$(GSFC80) between the USGS80 SV coefficients (Peddie & Fabiano 1982a) and the GSFC80 SV coefficients (Langel, Estes & Mead 1982)] is good enough a way to have estimates of these errors. Although this approach is probably not the best one, it is in good agreement with the results of Langel *et al.* (1986) who find errors on $\dot{b}_\gamma$ (qualified as 'crude estimations') which are slightly smaller than our estimates for degrees 1 to 4 but are comparable to those estimates for larger degrees. We also find a good agreement of the map of the difference between the USGS80 and CSFC80 models (not shown here) with the maps of Barker & Barraclough (1985) showing the error one can expect for any SV model because of the non-uniform geographic distribution of the magnetic measurements. Eventually, our evaluation is in good agreement with the conclusions of Barraclough (1990) and Lowes (1990). We therefore take our estimation as good enough for this study.

5.2 The part of the flow that is constrained by the present-day data

We now follow the same reasoning as in Section 4.4, but, instead of comparing $\langle R(\gamma, N_\alpha, N_\beta)\rangle$ with $\langle R(\gamma, N_\alpha)\rangle$, we will compare $\langle R(\gamma, N_\alpha, N_\beta)\rangle$ with $\langle \dot{b}_\gamma$(USGS80) $- \dot{b}_\gamma$(GSFC80)$\rangle$, the RMS value of the $\dot{b}_\gamma$(USGS80) $- \dot{b}_\gamma$(GSFC80) for a given degree n_γ of the SV, and conclude that if N_β is such that $\langle R(\gamma, N_\alpha, N_\beta)\rangle$ is smaller than $\dot{b}_\gamma$(USGS80) $- \dot{b}_\gamma$(GSFC80)$\rangle$, then the components of the flow of degree larger than N_β are not at all constrained by the data. Fig. 9(a) illustrates the study we made

Figure 9. See Section 5.2. As a function of n_γ, the SV degree. (a) The RMS value $\langle \dot{b}_\gamma$(USGS80) $- \dot{b}_\gamma$(GSFC80)$\rangle$ (open diamonds); the RMS value of the $\langle R(\gamma, N_\alpha = 13, N_\beta = 8)\rangle$ over the several possible m_γ (full triangles); same for the $\langle R(\gamma, N_\alpha = 13, N_\beta = 7)\rangle$ (open stars); same for the $\langle R(\gamma, N_\alpha = 13, N_\beta = 6)\rangle$ (open circles). (b) The RMS value $\langle \dot{b}_\gamma$(USGS80) $- \dot{b}_\gamma$(GSFC80)$\rangle$ (open diamonds); the RMS value $\langle \dot{b}_\gamma$(USGS80) $- \dot{b}_\gamma$(typical flow)$\rangle$, where the $\dot{b}_\gamma$(typical flow) are the coefficients of the SV created by the 'typical' flow—see text (open stars); the RMS value $\langle \dot{b}_\gamma$(USGS80) $- \dot{b}_\gamma$(second flow)$\rangle$, where the $\dot{b}_\gamma$(second flow) are the coefficients of the SV created by the second flow—see text (full triangles).

following this criterion. It clearly comes out that the degrees of the flow larger than 8 are not at all constrained by the present SV data.

In order to evaluate more precisely the accuracy of the components of the flow with degree 1 to 8, we will derive a first, explicit model of the flow. Because the 1980 Main Field is very well defined (thanks to MAGSAT data) we decided to make our computation for the year 1980. We assume the flow is geostrophic (we use the tangentially geostrophic basis). Although, as already mentioned, this is not a key point (see the remark at the end of the section), the following results will therefore preferentially apply to geostrophic motions. The calculation is very similar to those done by Gire & Le Mouël (1990) and

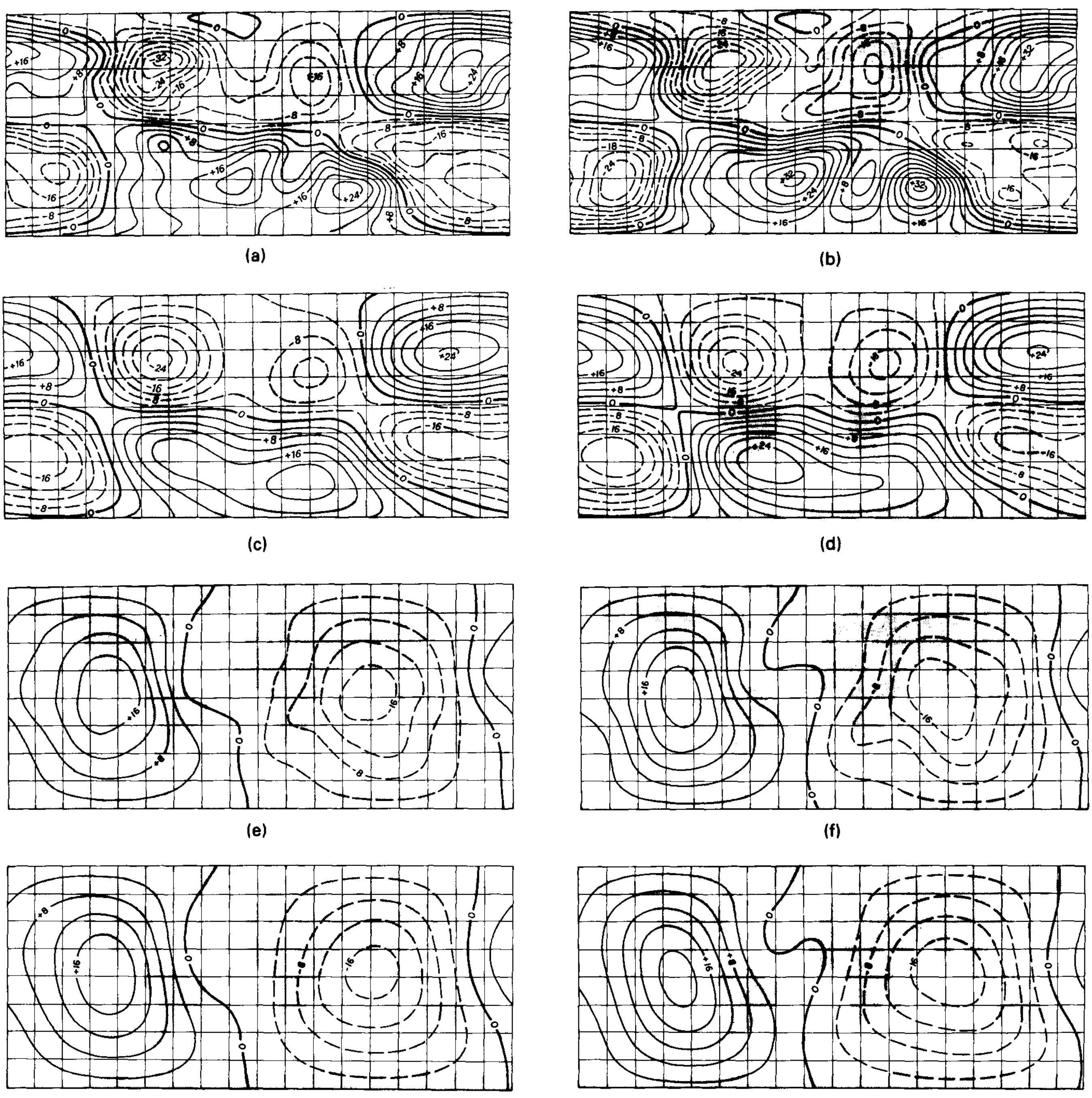


Figure 10. See Section 5.2. Scale: $10^{-4}\,\mathrm{rad}^2\,\mathrm{yr}^{-1}$. Parallels are shown every 20° between −80° and 80° and meridians are shown every 20°, Greenwich meridian being at the centre of the picture. Dashed lines for negative values, full lines for positive values. (a) Toroidal scalar of the 'typical' flow (see text). (b) Toroidal scalar of the second flow (see text). (c) Toroidal scalar of the 'typical' flow truncated at degree 4. (d) Toroidal scalar of the second flow truncated at degree 4. (e) Poloidal scalar of the 'typical' flow. (f) Poloidal scalar of the second flow. (g) Poloidal scalar of the 'typical' flow truncated at degree 4. (h) Poloidal scalar of the second flow truncated at degree 4.

Hulot, Le Mouël & Jault (1990):

(i) the flow is calculated up to degree 20, order 19;
(ii) the energy of the flow is required to converge in the way described in Section 4.1 [recall (27)] with $l = 2$;
(iii) the SV created by the calculated flow is required to fit the USGS80 SV model within the error previously defined; and
(iv) the MF model (up to degree 13) is taken from Cohen & Achache (1990).

Since we just proved that the components of the flow of degree larger than 8 cannot be determined, this flow is definitely not the flow occurring at the CMB: we can feel fairly confident about the lowest degrees components of the computed flow (within the limit we are going to explore next) whereas we can only say that the high-degree components are acceptable (i.e. complying with all our *a priori* requests). We will refer to this flow as a 'typical flow' (this is the flow we referred to in Sections 4.1, 4.2 and 4.3).

For the reasons developed above we argue that a second equally valid flow may be obtained by truncating *a priori* the flow at degree 8 and using another satisfying (within the error defined in Section 5.1) SV model. We decided to use the USGS80 SV model up to degree 6 and to set all the coefficients of degree 7 and 8 to 0 [the errors for these coefficients are of the same size as the coefficients themselves: compare Fig. 9(a) to Fig. 8(a)].

According to the preceding discussion, the difference between the SV created by this second flow (as well as the one created by the 'typical' flow) and the USGS80 SV should not be larger than the difference between the two SV models. Fig. 9(b) shows that this is indeed the case. Hence, comparing this second acceptable flow with the 'typical flow' gives us a hint of the accuracy we can expect for the flow at the CMB when using the available magnetic data.

Figure 10 shows the maps of the toroidal and poloidal scalar of both flows. To underline the very strong similarities existing between these flows for low-degree components, we also show the maps we obtain if we truncate them at degree 4. A more quantitative way of comparing these two flows is to look at their spectra and at the spectrum of their difference (Fig. 11). It appears that the energy of both flows are very similar and almost identical for degrees 1 to 4, and that the energy of the difference is always smaller than the energy of both flows (by a factor of about 10 in the case of degrees 1 to 4), which confirms what was seen on Fig. 10.

We therefore conclude that, whereas we cannot resolve the flow for degrees larger than 8, and despite the fact the accuracy is poor for degrees 5 to 8 (Fig. 11; this is especially true for the poloidal component, but it is a consequence of the geostrophic assumption that the poloidal component be less energetic than the toroidal component), the components of the flow of degrees 1 to 4 are fairly well known. More precisely, if we introduce $\sigma_s(n_\beta)$ and $\sigma_t(n_\beta)$, the RMS errors on the s_β and t_β, we have [recall (25)]

$$\sigma_s^2(n_\beta) \approx 0.1\frac{E_s(n_\beta)}{n_\beta(n_\beta+1)}, \qquad \sigma_t^2(n_\beta) \approx 0.1\frac{E_T(n_\beta)}{n_\beta(n_\beta+1)}, \qquad \text{for } n_\beta \in \{1, 2, 3, 4\}.$$

One should have this conclusion in mind when dealing with the different maps or figures a number of authors have produced for the flow at the CMB. In particular, the components of degree larger than 8 do not rely on the data but on *a priori* information. So, in order to compare two different flow models, one should truncate them at least at degree 8 if not at degree 4. Let us give two typical examples. A rather common way of testing the geostrophic assumption is to compute a flow using basic constraints (the convergence of the energy) and see whether it does or does not have the equator-no-crossing geostrophic property (Benton 1985). But this property must be tested on large scales: a local crossing cannot be considered as significant (Bloxham 1989). As a second example, we recall a recent study by Gire & Le Mouël (1990) who, in the purpose of establishing some symmetry properties, used among other arguments the fact that a part of the flow 'showed no tendency to converge' (see their Fig. 5). We can now state that this is the result of some kind of uncontrolled *a priori* constraint (in fact a constraint on the energy which was awkwardly imposed in the numerical programs). It is possible to make this part of the flow converge [see Fig.

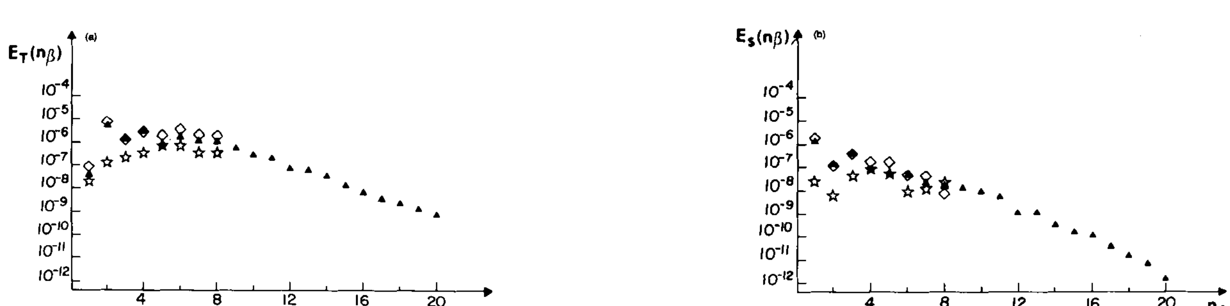


Figure 11. See Section 5.2. As a function of n_β, the flow degree, the energy [units in (rad yr^{-1})^2] of: the 'typical' flow—see text (full triangles); the second flow—see text (open squares); the difference between the typical and the second flow (open stars). (a) Toroidal components. (b) Poloidal components.

11(b) versus their Fig. 5]. We however insist that this does not question the main result of their study that the flow has some very interesting symmetry properties [confirmed and further studied by Hulot *et al.* (1990)].

REMARK

In the main part of this study, we assumed the flow is large scale with no *a priori* correlation between the different components nor any *a priori* constraint on the toroidal or poloidal part (recall Section 4.1). But one usually imposes some further constraints on the flow, as mentioned in the Introduction. These are the toroidal, the tangentially geostrophic or the steady motion constraint. It is thus of some importance to see whether these further constraints do or do not affect our study.

Let us assume the flow is purely toroidal. As a consequence the poloidal coefficients s_β vanish in (29). This is equivalent to the statement that the $\mathbf{J}'$ interaction integrals be zero, and amounts to divide the value given by (31c) by a factor of about $\sqrt{2}$. At the same time, to keep the same value for the *a priori* energy of the flow, $\langle u(n_\beta)\rangle$ has to be increased by the same factor. These two effects balance, so that nothing has to be changed in our final results [formulae (32) and (35)]. If we now assume that the flow is tangentially geostrophic, the situation seems more tricky since this assumption introduces correlations between the toroidal and the poloidal components of the flow. This case could be treated exactly. But taking advantage of the fact that, for large degrees, the tangentially geostrophic flow is nearly toroidal, it is easy to show that the results are very similar to those we obtained. As for the steady motion constraint, it amounts to using equation (3) at different epoches to constrain the steady flow. Our analysis can be applied to each epoch so that the formulae (32) and (35) still hold.

6 CONCLUSIONS

In this paper our concern was again to try and give some qualitative and quantitative answers to the question: what can be said about the flow at the CMB from magnetic data collected at the Earth's surface? The question is easy to state but covers a great variety of different aspects. One immediately thinks of the validity of the frozen flux theorem, of the adequacy of modelling the mantle as an insulator or of the fundamental ambiguity, three points that are at the very basis of the method. While not neglecting the importance of these three points, we could fear a greater obstacle might come from other important aspects of the problem such as the errors in the SV models or the less obvious errors linked to the unavoidable truncations of the MF and of the motion.

We showed that these two shortcomings impose some serious limitations on what can be said about this flow. On one hand, because the accuracy of the 1980 SV model coefficients gradually worsens with the degree to the point that coefficients with degree larger than 6 are actually unknown, the only components of the flow that can be trusted (within the frame of the usual hypothesis: frozen flux; insulating mantle; toroidal, tangentially geostrophic or steay flows) are those with degree less than 4 or 5. On the other hand, our study of the errors linked to the truncation of the MF showed that, had we known perfectly the SV, the flow could have been calculated for degrees as large as 12. This tends to prove that quite some improvement might be expected by increasing the quality of the SV models. More specifically, one can aim at reducing the errors in the SV models to the level of the errors linked to the truncation of the field. As one can see when comparing Fig. 8 to Fig. 9, the accuracy for degree 1 is already good enough and the accuracy for degree 2 only needs an improvement by a factor of 2 to meet this condition. On the contrary, a lot of improvement is still needed for larger degrees. In the case of degree 8 for instance, the aim would be to get an error as low as $0.15\,\mathrm{nT\,yr^{-1}}$ (at the Earth's surface). This is a very low value one might not be ever able to reach because essentially of the difficulties encountered in separating the external from the internal magnetic signals. But there is no doubt some improvement is possible in the future. Let us make some simple suggestions. The SV models built to apply the IGRF 1980 label were intended to be predictive. They suffer the noticeable inconvenience of being extrapolations of past data—for instance the USGS80 SV model has been constructed on the basis of 1976 to 1981 data in order to predict the field from 1980 to 1985 (Peddie & Fabiano 1982b). Such is not the case for models more specifically constructed for the study of the Earth's core (Langel *et al.* 1986; Bloxham & Jackson 1989) for which the authors interpolate rather than extrapolate the data. Unfortunately none of these models uses data more recent than 1981 so that they do not describe the 1980 SV better than the USGS80 model. Were they extended to more recent data, an improved 1980 SV model could certainly be constructed. As for future SV models, any mean of increasing the number of data (at best covering the Earth's surface) would be very valuable: magnetic satellites or an implemented array of observatories such as INTERMAGNET.

In any case, let us emphasize that, even in the present day situation, low-degree terms of the flow (those with degree less than, say, 5) are fairly well known (within the quoted basic hypothesis).

ACKNOWLEDGMENTS

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APPENDIX A

The aim of this appendix is to give a proof of formula (22b) (see text section 3.2). This formula requires that n_α is large compared to n_γ, which will be assumed in the following.

Recalling (12) (see section 2.2 of the text) and using the simple result:

$$cY_\gamma \nabla_H(\mathbf{X}_{\alpha\beta}) = c\nabla_H(Y_\gamma \mathbf{X}_{\alpha\beta}) - \mathbf{S}_\gamma . \mathbf{X}_{\alpha\beta}$$

together with:

$$\int\int_S \nabla_H(Y_\gamma \mathbf{X}_{\alpha\beta})\,dS = 0$$

$X_{\alpha\beta}$ being either $Y_\alpha \mathbf{S}_\beta$ or $Y_\alpha \mathbf{T}_\beta$, we readily transform the expression of $J(\beta,\alpha,\gamma)$ and $J'(\beta,\alpha,\gamma)$ into:

$$\begin{cases} J(\beta,\alpha,\gamma) = \int\int_S Y_\alpha \mathbf{S}_\gamma . \mathbf{S}_\beta dS \\ J'(\beta,\alpha,\gamma) = \int\int_S Y_\alpha \mathbf{S}_\gamma . \mathbf{T}_\beta dS \end{cases} \tag{A1}$$

Considering the vector $Y_\alpha \mathbf{S}_\gamma$ (α and γ being given), we see from (A1) that:

$$Y_\alpha \mathbf{S}_\gamma = \sum_\beta \left(J(\beta,\alpha,\gamma) \frac{\mathbf{S}_\beta}{||\mathbf{S}_\beta||^2} + J'(\beta,\alpha,\gamma) \frac{\mathbf{T}_\beta}{||\mathbf{T}_\beta||^2} \right) \tag{A2}$$

where the summation is done for all possible β when α and γ are given.

Recalling that the selection rules (15) and (16) imply that $n_\beta \sim n_\alpha$ when n_α becomes large, so that from (7) (see section 2.1 in the text), we get:

$$||\mathbf{S}_\beta||^2 = ||\mathbf{T}_\beta||^2 \sim 2\pi n_\alpha$$

it then follows from taking the scalar square of (A2) that:

$$\int\int_S (Y_\alpha \mathbf{S}_\gamma)^2 dS \sim \frac{1}{2\pi n_\alpha} \sum_\beta (J(\beta,\alpha,\gamma)^2 + J'(\beta,\alpha,\gamma)^2)$$

If we now keep n_α and γ fixed while summing over the m_α and i_α we obtain (recall definition (21) in section 3.2 of the text):

$$\sum_{\substack{m_\alpha=0 \\ i_\alpha \in \{c,s\}}}^{n_\alpha} \int\int_S (Y_\alpha \mathbf{S}_\gamma)^2 dS \sim \frac{1}{2\pi n_\alpha} (N(n_\alpha,\gamma) + N'(n_\alpha,\gamma)) < J(n_\alpha,\gamma) >^2$$

Using the addition theorem concerning the $P_n^m(\cos\theta)$ in a special case, we have:

$$\sum_{\substack{m_\alpha=0 \\ i_\alpha \in \{c,s\}}}^{n_\alpha} Y_\alpha^2(\theta,\phi) = \sum_{m_\alpha=0}^{n_\alpha} P_\alpha^2(\cos\theta) = P_{n_\alpha}(1) = 1$$

So (Recall (7)):

$$\sum_{\substack{m_\alpha=0 \\ i_\alpha \in \{c,s\}}}^{n_\alpha} \int\int_S (Y_\alpha \mathbf{S}_\gamma)^2 dS = \int\int_S \mathbf{S}_\gamma^2 dS = 4\pi \frac{n_\gamma(n_\gamma+1)}{2n_\gamma+1}$$

Eventually, $N(n_\alpha,\gamma)$ and $N'(n_\alpha,\gamma)$ have already been computed in the text (see section 4.2); for large n_α, $N(n_\alpha,\gamma) + N'(n_\alpha,\gamma)$ is $2n_\alpha(2n_\gamma+1)$ if $m_\gamma = 0$ and $4n_\alpha(2n_\gamma+1)$ if $m_\gamma \neq 0$. Then:

$$< J(n_\alpha,\gamma) > \sim \frac{\sqrt{n_\gamma(n_\gamma+1)}}{n_\gamma + \frac{1}{2}} \times \begin{cases} \pi \; if \; m_\gamma = 0 \\ \frac{1}{\sqrt{2}}\pi \; if \; m_\gamma \neq 0 \end{cases}$$

APPENDIX B

The actual values of the b_α depend on the origin of the longitude (currently Greenwich meridian). This leads us to investigate whether the assumption that the b_α coefficients may be seen as independently drawn lots of a random variable $b(n_\alpha)$ also depends on this origin.

Let us suppose we now describe the M.F. using $\phi = \phi_0$ as the new longitude's origin. b'_α are the corresponding coefficients. One has:

$$\begin{cases} b'^{m_\alpha c}_{n_\alpha} = \ \ b^{m_\alpha c}_{n_\alpha} \cos m_\alpha \phi_0 + b^{m_\alpha s}_{n_\alpha} \sin m_\alpha \phi_0 \\ b'^{m_\alpha s}_{n_\alpha} = -b^{m_\alpha c}_{n_\alpha} \sin m_\alpha \phi_0 + b^{m_\alpha s}_{n_\alpha} \cos m_\alpha \phi_0 \end{cases}$$

Since $E[b(n_\alpha)] = 0$, we have:

$$E\left[b'^{m_\alpha c}_{n_\alpha}\right] = E\left[b'^{m_\alpha s}_{n_\alpha}\right] = 0$$

The lack of correlations between the b_α implies the lack of correlations between the b'_α, even between $b'^{m_\alpha c}_{n_\alpha}$ and $b'^{m_\alpha s}_{n_\alpha}$:

$$\begin{aligned} E\left[b'^{m_\alpha s}_{n_\alpha} b'^{m_\alpha c}_{n_\alpha}\right] &= E\left[\left(b^{m_\alpha s}_{n_\alpha}\right)^2 - \left(b^{m_\alpha c}_{n_\alpha}\right)^2\right] \cos m_\alpha \phi_0 \sin m_\alpha \phi_0 \\ &\quad + E\left[b^{m_\alpha s}_{n_\alpha} b^{m_\alpha c}_{n_\alpha}\right](\cos^2 m_\alpha \phi_0 - \sin^2 m_\alpha \phi_0) \\ &= 0 \end{aligned}$$

Eventually, there is no difficulty in proving that:

$$E[(b'_\alpha)^2] = \langle b(n_\alpha) \rangle^2$$

Then the b'_α follow a statistic similar to the one obeyed by the b_α.

APPENDIX C

This appendix aims at proving that assuming (33) and (34) rather than assuming the more handy statement $E[b(n_\alpha)] = E[u(n_\beta)] = 0$ does not affect the result (32) of section 4.2. We introduce the following simple notations: X_{Old} will refer to the unknown X when using the assumptions of the text (i.e. $E[b(n_\alpha)] = E[u(n_\beta)] = 0$), whereas X_{New} will refer to the same unknown X but using the assumptions of this appendix (i.e. (33) and (34)).

Our process is simple. We will show that $\Delta^2_R(\gamma, N_\alpha)$ defined by:

$$\Delta^2_R(\gamma, N_\alpha) = \frac{\langle R(\gamma, N_\alpha) \rangle^2_{New} - \langle R(\gamma, N_\alpha) \rangle^2_{Old}}{\langle R(\gamma, N_\alpha) \rangle^2_{Old}} \qquad (C1)$$

remains small enough for the order of magnitude of $\langle R(\gamma, N_\alpha) \rangle^2_{New}$ to remain the same as the order of magnitude of $\langle R(\gamma, N_\alpha) \rangle^2_{Old}$.

In this appendix we assume again $n_\alpha \sim n_\beta \gg n_\gamma$. Taking into account (33) and (34), the mathematical expectation of $B(\gamma, n_\alpha)$ is now:

$$E[B(\gamma, n_\alpha)]_{New} = \epsilon_b \epsilon_u F(\gamma, n_\alpha)$$

with:

$$F(\gamma, n_\alpha) = \frac{n_\gamma(n_\alpha+1)\epsilon_b(n_\alpha)}{||\mathbf{T}_\gamma||^2} \frac{\epsilon_b(n_\alpha)}{\epsilon_b} < b(n_\alpha) > \sum_{n_\beta} \frac{\epsilon_u(n_\beta)}{\epsilon_u} < u(n_\beta) > (N(n_\beta, n_\alpha, \gamma) + N'(n_\beta, n_\alpha, \gamma)) \overline{J(n_\beta, n_\alpha, \gamma)}$$

where the summation is performed on all the n_β allowed by the selection rules (15) and (16) when (γ, n_α) is given, and where $\overline{J(n_\beta, n_\alpha, \gamma)}$ is the mean value of the $N(n_\beta, n_\alpha, \gamma)$ possible $J(\beta, \alpha, \gamma)$ coefficients and $N'(n_\beta, n_\alpha, \gamma)$ possible $J'(\beta, \alpha, \gamma)$ coefficients when $(n_\beta, n_\alpha, \gamma)$ is given.

Before going any further, we need to describe the behaviour of the $\overline{J(n_\beta, n_\alpha, \gamma)}$ when n_α becomes large compared to n_γ. Fig.12 gives a numerical illustration of this behaviour. Each of the two drawings corresponds to a given γ (in fact we chose the same γ as for Fig.4 for comparison) and we plotted all $\overline{J(n_\beta, n_\alpha, \gamma)}$ mean values allowed by the selection rules (15) and (16) (i.e. $|n_\alpha - n_\gamma| \le n_\beta \le n_\alpha + n_\gamma$) for each degree n_α up to degree $25 - n_\gamma$. Despite the fact that Fig.4 is showing a somehow even distribution of the $J(\beta, \alpha, \gamma)$ and $J'(\beta, \alpha, \gamma)$ coefficients about the value 0, it is obvious that there is not such a distribution for the coefficients involved in the "partial" (recall that n_β is fixed in this mean value) mean value $\overline{J(n_\beta, n_\alpha, \gamma)}$. This makes it impossible to derive as remarkable a result for the $\overline{J(n_\beta, n_\alpha, \gamma)}$ as the one ((22)) we derived for the $< J(n_\alpha, \gamma) >$. An upper bound for the $\overline{J(n_\beta, n_\alpha, \gamma)}$ is given by the R.M.S. value $< J(n_\beta, n_\alpha, \gamma) >$ of the J and J' coefficients involved in the calculation of the mean value $\overline{J(n_\beta, n_\alpha, \gamma)}$:

$$|\overline{J(n_\beta, n_\alpha, \gamma)}| \le < J(n_\beta, n_\alpha, \gamma) > \qquad (C2)$$

Returning to $F(\gamma, n_\alpha)$, because the sign of $\epsilon_u(n_\beta)$ is a random function of n_β, an estimate

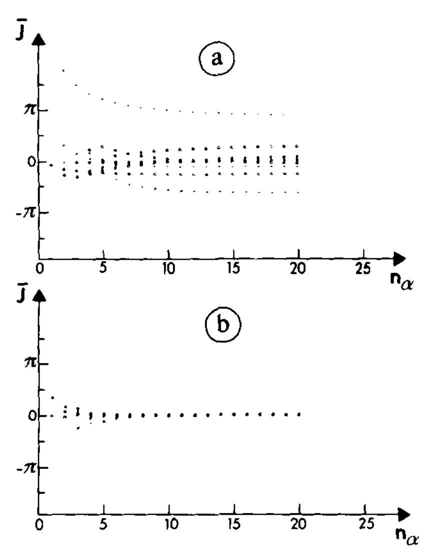

Figure 12. See Appendix C. Mean values $\bar{J}(n_\beta, n_\alpha, \gamma)$ of the J and J′ interaction integrals when $(n_\beta, n_\alpha, \gamma)$ is given, as a function of n_α, the degree of the Main Field, for: (a) $n_\gamma = 5$, $m_\gamma = 4$, i_γ = cosine; (b) $n_\gamma = 5$, $m_\gamma = 4$, i_γ = sine.

of the order of magnitude of $F(\gamma,n_\alpha)$ is its R.M.S. value $<F(\gamma,n_\alpha)>$:

$$<F(\gamma,n_\alpha)> = \frac{n_\gamma(n_\alpha+1)}{\|T_\gamma\|^2}<b(n_\alpha)>\times$$

$$\left(\sum_{n_\beta}<u(n_\beta)>^2(N(n_\beta,n_\alpha,\gamma)+N'(n_\beta,n_\alpha,\gamma))^2\overline{J(n_\beta,n_\alpha,\gamma)^2}\right)^{\frac{1}{2}} \qquad (C3)$$

As for the sign of $F(\gamma,n_\alpha)$, it can also be considered as a random function of n_α with no a priori correlation with $\epsilon_b(n_\alpha)$, so we may state:

- the sign of $E[B(\gamma,n_\alpha)]_{New}$ is a random function of n_α.

- $O\{E[B(\gamma,n_\alpha)]_{New}\} \approx \epsilon_u\epsilon_b<F(\gamma,n_\alpha)>$.

$O\{X\}$ being the order of magnitude of the value X.

Starting with (C3), recalling (C2), noting that:

$$N(n_\beta,n_\alpha,\gamma)+N'(n_\beta,n_\alpha,\gamma) \le \frac{2n_\alpha}{1+\delta_{m_\gamma,0}}+1$$

and comparing to (31a) (see section 4.2), we readily have:

$$<F(\gamma,n_\alpha)> \le \sqrt{\frac{2n_\alpha}{1+\delta_{m_\gamma,0}}+1}<B(\gamma,n_\alpha)>_{Old}$$

Since we are only interested in order of magnitude calculations, we may therefore conclude that:

$$O\{E[B(\gamma,n_\alpha)]_{New}\} < \sqrt{\frac{2}{1+\delta_{m_\gamma,0}}}\sqrt{n_\alpha}\,\epsilon_u\epsilon_b<B(\gamma,n_\alpha)>_{Old} \qquad (C4)$$

Let us now evaluate the order of magnitude of the relative change of $<B(\gamma,n_\alpha)>^2$, $\Delta_B(\gamma,n_\alpha)^2$, defined by:

$$\Delta_B^2(\gamma,n_\alpha) = \frac{<B(\gamma,n_\alpha)>^2_{New}-<B(\gamma,n_\alpha)>^2_{Old}}{<B(\gamma,n_\alpha)>^2_{Old}}$$

We easily obtain:

$$\Delta_B^2(\gamma,n_\alpha) = \frac{E[B(\gamma,n_\alpha)]^2_{New}}{<B(\gamma,n_\alpha)>^2_{Old}} - \epsilon_b^2\epsilon_u^2$$

Using (C4) and recalling that $n_\alpha \gg 1$, we then get:

$$O\{\Delta_B^2(\gamma,n_\alpha)\} < \frac{2}{1+\delta_{m_\gamma,0}}n_\alpha\epsilon_u^2\epsilon_b^2 \qquad (C5)$$

It remains to estimate the order of magnitude of the relative change $\Delta_R^2(\gamma,N_\alpha)$ defined by (C1). We have:

$$<R(\gamma,N_\alpha)>^2_{New} = \sum_{n_\alpha=N_\alpha+1}^{\infty}<B(\gamma,n_\alpha)>^2_{New}+\left(\sum_{n_\alpha=N_\alpha+1}^{\infty}E[B(\gamma,n_\alpha)]_{New}\right)^2-\sum_{n_\alpha=N_\alpha+1}^{\infty}E[B(\gamma,n_\alpha)]^2_{New}$$

But:

$$\sum_{n_\alpha=N_\alpha+1}^{\infty} < B(\gamma,n_\alpha)>^2_{New} = < R(\gamma,N_\alpha)>^2_{Old} + \sum_{n_\alpha=N_\alpha+1}^{\infty} \Delta_B^2(\gamma,n_\alpha) < B(\gamma,n_\alpha)>^2_{Old}$$

So:

$$\Delta_R^2(\gamma,N_\alpha) = \frac{1}{< R(\gamma,N_\alpha)>^2_{Old}} \Bigg(\sum_{n_\alpha=N_\alpha+1}^{\infty} \Delta_B^2(\gamma,n_\alpha) < B(\gamma,n_\alpha)>^2_{Old} + \left(\sum_{n_\alpha=N_\alpha+1}^{\infty} E[B(\gamma,n_\alpha)]_{New} \right)^2 - \sum_{n_\alpha=N_\alpha+1}^{\infty} E[B(\gamma,n_\alpha)]^2_{New} \Bigg) \qquad (C6)$$

The sign of $E[B(\gamma,n_\alpha)]_{New}$ is a random function of n_α, so that:

$$O\left\{ \left(\sum_{n_\alpha=N_\alpha+1}^{\infty} E[B(\gamma,n_\alpha)]_{New} \right)^2 \right\} \approx O\left\{ \sum_{n_\alpha=N_\alpha+1}^{\infty} E[B(\gamma,n_\alpha)]^2_{New} \right\}$$

Recalling (C4) and (C5), we also have:

$$\left. \begin{array}{l} O\left\{ \dfrac{\sum_{n_\alpha=N_\alpha+1}^{\infty} \Delta_B^2(\gamma,n_\alpha) < B(\gamma,n_\alpha)>^2_{Old}}{< R(\gamma,N_\alpha)>^2_{Old}} \right\} \\ O\left\{ \dfrac{\left(\sum_{n_\alpha=N_\alpha+1}^{\infty} E[B(\gamma,n_\alpha)]_{New} \right)^2}{< R(\gamma,N_\alpha)>^2_{Old}} \right\} \end{array} \right\} < \frac{2}{1+\delta_{m_\gamma,0}} n_\alpha \epsilon_u^2 \epsilon_b^2$$

Hence the three terms of the left hand side of (C6) have orders of magnitude bounded by a common value, which leads to the result:

$$O\{\Delta_R(\gamma,N_\alpha)^2\} < \frac{6}{1+\delta_{m_\gamma,0}} \epsilon_u^2 \epsilon_b^2 S(N_\alpha) \qquad (C7)$$

with:

$$S(N_\alpha) = \frac{\sum_{n_\alpha=N_\alpha+1}^{\infty} n_\alpha B^2(n_\alpha)}{\sum_{n_\alpha=N_\alpha+1}^{\infty} B^2(n_\alpha)}$$

The assumptions of the text (see section 4.1) lead to a practical formula for $B(n_\alpha)$ (recall (23), (27) and (31c)), given by:

$$B(n_\alpha) = \frac{e^{-\frac{k}{2}n_\alpha}}{n_\alpha^{\frac{1}{2}(1+l)}} \quad , \quad k \approx 0.14 \quad , \quad l > 1$$

It is worth noting that even if we take $k = 0$ (corresponding to a white spectrum for the field at the C.M.B.), the simple requirement $l > 1$ (corresponding to the requirement that the energy of the flow is converging) ensures the existence of $S(N_\alpha)$. For the three cases we studied in the text (section 4.3), we obtain respectively:

$$S(N_\alpha = 13) \approx 17.7 \quad for \quad l = 1.1$$

$$S(N_\alpha = 13) \approx 17.0 \quad for \quad l = 2$$

$$S(N_\alpha = 13) \approx 16.4 \quad for \quad l = 3$$

showing again a low sensitivity to the exact value of l. We will keep the value $S(N_\alpha = 13) \approx 17$. Taking $\epsilon_a = 0.3$ and $\epsilon_b = 0.2$ (see text), (C7) leads to the quantitative result:

$$O\{\Delta_R(\gamma, N_\alpha = 13)^2\} < \frac{1}{1 + \delta_{m_\gamma, 0}} 0.4$$

Hence $< R(\gamma, N_\alpha = 13) >_{New}$ is not different from $< R(\gamma, N_\alpha = 13) >_{old}$ by more than 20%.