Recurrence relations in m x 3 scaffolding and globe resistor networks

Frédéric Perrier, Frédéric Girault

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Frédéric Perrier* **and Frédéric Girault**

Université de Paris, Institut de Physique du Globe de Paris, CNRS, F-75005 Paris, France

* Author to whom any correspondence should be addressed.

E-mail: perrier@ipgp.fr **and** girault@ipgp.fr

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Abstract

Resistor networks are popular because they offer solvable models of transport between connected discrete points and can represent natural or artificial systems such as mycorhizzal networks or carbon composite chains. When the connectivity pattern is repeated, two-point resistances can be expressed by recurrence relations. Here, we illustrate this approach in the case of three-dimensional $m \times 3$ scaffolding and globe networks, characterized by a repeated pattern along a three-fold invariant axis. We show that a first set of recurrence relations follows from three-fold invariance and Kennelly's Y-Δ transform, providing the two-point resistance between any pair of neighbouring nodes, including the case of infinite networks. Using van Steenwijk's method, a second set of recurrence relations is obtained between non-neighbouring nodes. Numerous explicit expressions are thus derived using elementary methods, which can be compared with the results of numerical codes or novel integration methods such as Tan's recursion-transform. Having at hand general properties of these networks is useful to evaluate their capacity of representing natural or industrial systems.

1. Introduction

Networks represent matter, not by smooth continuous functions, but by a discrete number of connected points. Since their properties can be solved analytically or numerically, networks play an important role in the approach of complexity in contemporary physics [1]; they are, for example, efficient models for transport problems [2] or adaptive decision processes [3]. Resistor networks, considered since the nineteenth century [4], provide a remarkable introduction to the generic properties of networks, and can be used to practice general methods, such as the Y-Δ transform [5], also known as Kennelly's theorem, or van Steenwijk's method [6]. Using these elementary methods, the resistance between any pair of nodes (two-point resistance), can be calculated exactly in numerous networks [6–8].

Among resistor networks, those with a repeating pattern of nodes and connections are being considered frequently in current research, for example to analyze the response of long chains of electronic components [9], or to model fibrous matter such as carbon composites [10]. Repeated patterns can also result from a growth mechanism or represent a reciprocal system in a packing problem in the context of granular matter [11]. The iteration of network units, even at a simple level, leads to recursive relations and, thus, generates scale-free (fractal) objects [12] and also provides graphical representations of concepts in number theory such as the expression of irrational numbers as continued fraction [13].

Resistor models are now applied routinely in rock physics [2, 14] or geophysical data interpretation [15–17], using numerical methods. Exact mathematical methods have also been developed such as the resistance sum rules [18], Green functions [19–22] and the Laplacian matrix approach [23], which was recently revisited to propose the direct recursion-transform (RT) method [24–30]. The RT method was originally proposed by Tan in 2011 [25, 27, 29] to be able to solve resistor networks configurations with arbitrary boundaries, which are important in practical situations. In this method, the potential difference between arbitrary nodes, given injection of current between two other arbitrary nodes, relies on the diagonalization of one single matrix and is

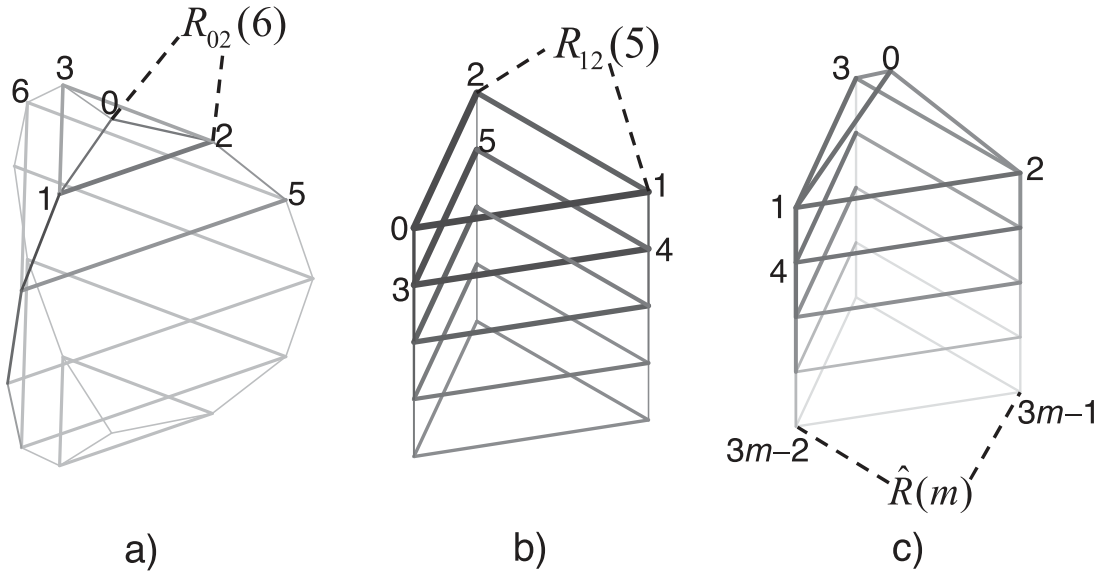

Figure 1. (a) 6×3 globe network (five triangular frames perpendicular to the three-fold axis). (b) 5×3 scaffolding or cylinder network (five triangular frames perpendicular to the three-fold axis). (c) 5×3 tower network (5×3 scaffolding with a top node). The frames perpendicular to the three-fold axis are all equal to b, and all resistors linking the frames or linking the top and bottom to polar nodes, when present, are equal to 1.

expressed as a generic sum. The general expression of the two-point resistance follows as a particular case [27], and various networks such as cobweb, fan, globe [25, 26, 28, 29] or triangular repeating networks [30] were completely solved in this manner. While, at the beginning of the development of the RT method [24, 25], the general expressions of the solution sums appeared difficult to handle, recently, easy to use explicit equations have been made available for repeating triangular or square networks [29]. In the presence of iterative patterns, however, the two-point resistances can be calculated in a simpler manner using recurrence relations, independently of the RT method. This is for example the case for ladders, which can be solved analytically [31, 32], and produce rich resonance structures when replacing some resistors with frequency-dependent LC impedances. Thus, it appeared interesting to compare the general exact solutions given by the RT method in situations where recurrence relations can be obtained directly.

In this paper, we concentrate on repeating three-dimensional resistor networks having a three-fold invariant axis, the $m \times 3$ globe network and its variants, the $m \times 3$ scaffolding and tower networks. In these cases, the two-point resistances can be calculated using elementary methods only. First, we introduce these networks in details and restate general properties resulting from the three-fold invariance and relations around corner nodes with three legs [33]. Then, we derive a first set of recurrence relations using the Y-Δ transform (Kennelly's theorem) and we obtain explicit expression for the two-point resistance between any pair of neighbouring nodes. Then, using van Steenwijk's method, a second set of recurrence relations is obtained and is used to infer the resistance between non-neighbouring nodes. Finally, we make a comparison with expressions obtained from the RT method [29] and we illustrate one application in terms of frequency-dependent impedance. The relevance of the approach and potential generalizations are discussed in the conclusion.

2. Presentation of three iterated three-fold invariant networks

Consider the $m \times 3$ globe network [24, 26] (figure 1(a)). It has 3-fold symmetry around a central axis. It is made of $m - 1$ horizontal frames of three equal resistors b and each frame is connected to the frame above and below, and for the top and bottom frames to the poles, by unity resistors. The top node (north pole) is labelled 0 and the bottom (south pole) is labelled $3(m - 1) + 1$. The nodes of the horizontal frames are numbered from the north pole down to the south pole as shown in figure 1(a).

While the nodes of globe networks are usually placed on a sphere [24, 26], the $m \times 3$ globe network is topologically equivalent to a pile of $m - 1$ equal triangular frames with poles, and this is the situation that we consider in the following. Thus, we construct the $m \times 3$ globe network from the $(m - 1) \times 3$ cylinder network (figure 1(b)). In the $m \times 3$ cylinder network, defined as a scaffolding (or a pile) of m frames, the nodes of the top frame are labelled 0, 1 and 2 and so on (figure 1(b)), with the last node being $3m - 1$. Taking an $(m - 1) \times 3$ cylinder network, we first add a top pole (figure 1(c)), a network referred to previously as the $(m - 1) \times 3$ tower network [8].

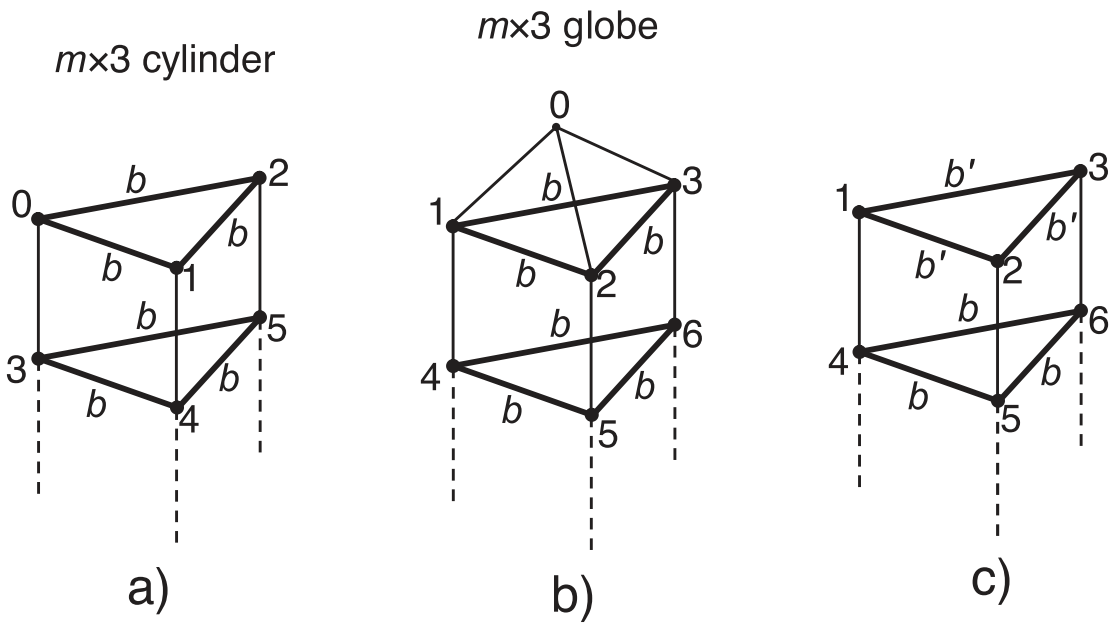

Figure 2. Upper parts of the $m \times 3$ cylinder (a) or globe (b) networks. When the top node of the globe network is removed by applying the Y to Δ transform, a cylinder network is obtained (c), but with a different resistor b' for the top frame instead of b.

Then, we turn it upside down, and add another pole, and thus obtain the $m \times 3$ globe network. When proceeding in this way, we obtain explicit relations simultaneously for the $m \times 3$ scaffolding and globe networks. The two-point resistance between nodes i and j in the $m \times 3$ cylinder network is noted $R_{ij}(m)$, and is noted $\widehat{R}_{ij}(m)$ in the $m \times 3$ globe network. The base resistance of the $m \times 3$ tower network is noted $\hat{R}(m)$, corresponding, for example, to the resistance between nodes $3m - 2$ and $3m - 1$ (figure 1(c)).

In a symmetrical network, two-point resistances are related to each other. It is useful to summarize the relations resulting from symmetries to avoid unnecessary calculations. In the present case, we first have three-fold invariance around the central axis, which leads to obvious relations such as $\widehat{R}_{01} = \widehat{R}_{02} = \widehat{R}_{03}$ (globe network, figure 1(a)) or $R_{01} = R_{12} = R_{20}$ (cylinder network, figure 1(b)). Another trivial consequence of the three-fold invariance is that the resistance between the two poles is $m/3$ (three lines of m unit resistors in series, connected in parallel). Less immediate relations, in the $m \times 3$ globe network, result from the fact that the poles of the $m \times 3$ globe network satisfy the conditions of the cubic corner theorem [8], and we have, for node 0:

$$\widehat{R}_{01} = \frac{1 + \widehat{R}_{12}}{3}, \tag{1}$$

with a similar relation for the south pole node.

Consider now the terminal nodes 0, 1 and 2 in the $m \times 3$ cylinder network (figure 2(a)). They satisfy the conditions of another corner theorem [8]. Indeed, these nodes are connected to three branches and there is a plane of symmetry. In the following, we need the most general version of the relations that result from this symmetry (two-plus-one legs corner theorem, see appendix A). In addition, because of the three-fold invariance, we have $R_{01} = R_{12}$. Thus, using equation (A11), we can express R_{03} and R_{04} in terms of R_{01}:

$$\begin{cases} R_{03} = \dfrac{3R_{01} - b(2 - b)}{b^2} \\ R_{04} = \dfrac{(3 + 3b + b^2)R_{01} - b(2 + b)}{b^2} \end{cases}. \tag{2}$$

Similar relations also exist for the nodes of the terminal frames of the $m \times 3$ globe network. Indeed, node 0 can be eliminated using Kennelly's theorem [5]. When one node K is connected to three nodes i with conductance c_i, then all connections from nodes i to K can be replaced with connections ij with conductance c_{ij} such that $c_{ij} = c_i c_j / \sum_k c_k$ (Y to Δ or star to triangle transform). When the three connections have the same resistance a, then the resistance of each replacing ij resistor is $3a$. Thus, the three branches from node 0 to nodes 1, 2 and 3 in the $m \times 3$ globe network, can be replaced by a resistor of value 3, thus nodes 1, 2 and 3 are now connected by a resistance b and 3 in parallel, namely a resistance $b' = 3b/(3 + b)$ (figure 2(c)). Now, we are in the situation of equation (2), replacing b by b':

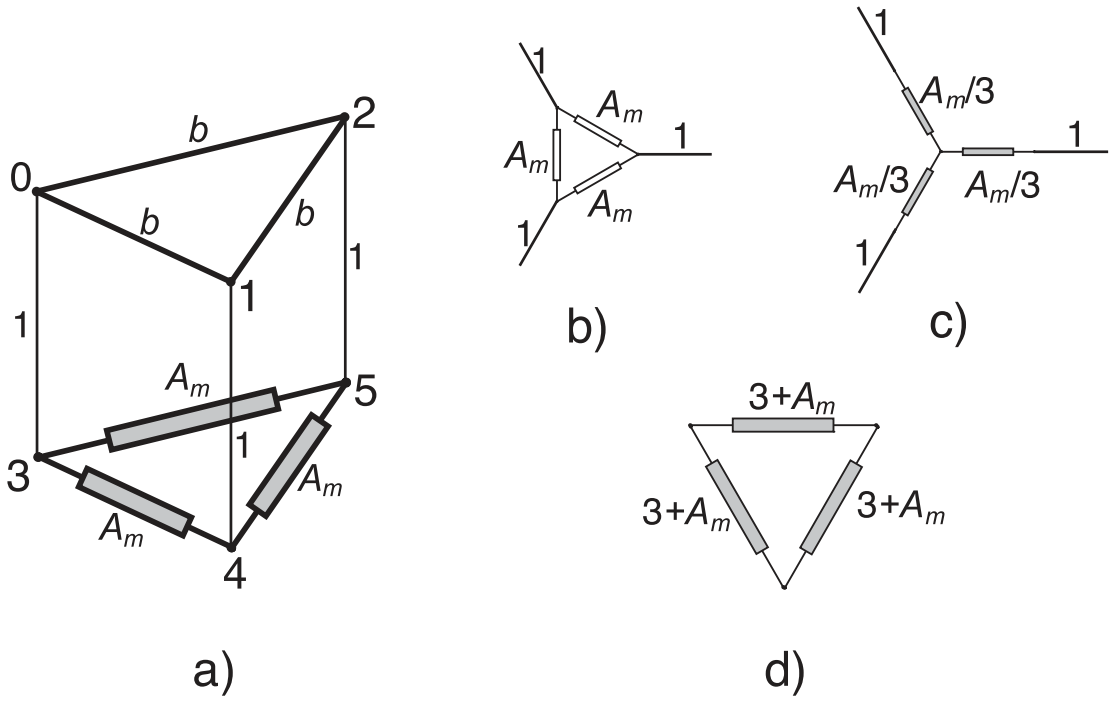

Figure 3. (a) Equivalent diagram of the top six nodes of the $m \times 3$ cylinder network. (b) Diagram showing the circuit below nodes 0, 1 and 2 in figure 3(a). (c) After application of the Δ to Y transform in the central part of figure 3(b). (d) After application of the Y to Δ transform to the Y circuit of figure 3(c).

$$\begin{cases} \widehat{R}_{14} = \dfrac{3\widehat{R}_{12} - b'(2-b')}{b'^2} = \dfrac{(3+b)^2\widehat{R}_{12} - b(6-b)}{3b^2} \\ \widehat{R}_{15} = \dfrac{(3+3b'+b'^2)\widehat{R}_{12} - b'(2+b')}{b'^2} = \dfrac{(9+15b+7b^2)\widehat{R}_{12} - b(6+5b)}{3b^2} \end{cases}. \tag{3}$$

Consequently, for the terminal nodes of the considered networks, it is sufficient to calculate R_{01} and $\widehat{R}_{12}$. In the following, we derive recurrence relations to obtain these resistances to any order m.

3. Recurrence relations for the terminal resistances

The considered networks are periodic or repeating networks, and are therefore handled by recurrence methods [9–13, 29, 30]. Let us consider the $m \times 3$ cylinder network and assume we know $R_{01}(m)$. Because of the three-fold rotational invariance, the equivalent diagram of the top three nodes is a triangle of equal resistances A_m, and we have $R_{01}(m) = 2A_m/3$ (A_m in parallel with $2A_m$). Now we add one frame of three resistors b and the vertical connections to the previous network (figure 3(a)). This new network is the $(m+1) \times 3$ cylinder network. To calculate the resistance between the new nodes 0 and 1 (figure 3(a)), we proceed as follows. First, we calculate the equivalent circuit connecting nodes 0, 1 and 2 to the bottom part, namely the unity resistors legs connected to the previous frame with resistances A_m. This circuit, shown in figure 3(b), can be reduced, first, by applying a Δ to Y transform at the centre. The central triangle is thus replaced by a star of $A_m/3$ to a central node (figure 3(c)). The whole circuit is now a star with three equal branches of resistance $1 + A_m/3$. Applying now a Y to Δ transform, we obtain a frame of three equal resistors $3 + A_m$ (figure 3(d)). The complete equivalent circuit between nodes 0, 1 and 2 is therefore a frame of three equal resistors $3 + A_m$ in parallel with b. Consequently, $A_{m+1} = b(3 + A_m)/(b + 3 + A_m)$ and:

$$R_{01}(m+1) = \frac{2}{3}A_{m+1} = 2b\frac{2 + R_{01}(m)}{6 + 2b + 3R_{01}(m)}. \tag{4}$$

This recurrence relation has numerous consequences. First, we infer that, at large values of m, $R_{01}(m)$ converges to the value $R_{01}(\infty)$, which satisfies:

$$R_{01}(\infty) = 2b\frac{2 + R_{01}(\infty)}{6 + 2b + 3R_{01}(\infty)}, \tag{5}$$

which is an equation of degree 2 with only one positive solution:

$$R_{01}(\infty) = \sqrt{1 + \frac{4b}{3}} - 1. \tag{6}$$

Using equation (2), we also have the limits for the other two edge resistances:

$$\begin{cases} R_{03}(\infty) = \dfrac{3R_{01}(\infty) - b(2-b)}{b^2} = \dfrac{3}{b^2}\sqrt{1 + \dfrac{4b}{3}} - \dfrac{3}{b^2} - \dfrac{2}{b} + 1 \\ R_{04}(\infty) = \dfrac{(3 + 3b + b^2)R_{01}(\infty) - b(2+b)}{b^2} = \dfrac{3 + 3b + b^2}{b^2}\left(\sqrt{1 + \dfrac{4b}{3}} - 1\right) - \dfrac{2}{b} - 1 \end{cases}. \tag{7}$$

The limit of the terminal resistance of the $m \times 3$ globe network also follows. Indeed, we take the $(m-1) \times 3$ tower network, turn it upside down, and the top nodes are equivalent to a frame of three resistors $\hat{A}_{m-1} = 3\hat{R}(m-1)/2$. Then, we add one pole and three legs, equivalent to a triangular frame of resistors 3, to add in parallel to $\hat{A}_{m-1}$, hence:

$$\widehat{R}_{12}(m) = \frac{2}{3}\frac{3\hat{A}_{m-1}}{3 + \hat{A}_{m-1}} = \frac{2\hat{R}(m-1)}{2 + \hat{R}(m-1)}. \tag{8}$$

Since $\hat{R}_m$ also follows the recurrence relation of equation (4), then $\hat{R}(\infty) = R_{01}(\infty)$, and equation (8) gives, at large m:

$$\widehat{R}_{12}(\infty) = \frac{2\hat{R}(\infty)}{2 + \hat{R}(\infty)} = \frac{3}{b}\left(1 + \frac{2b}{3} - \sqrt{1 + \frac{4b}{3}}\right). \tag{9}$$

Using the expression for $\widehat{R}_{12}(\infty)$ from equation (8), which we put into equations (1) and (3), we obtain the limits of the other top resistances of the $\infty \times 3$ globe network:

$$\begin{cases} \widehat{R}_{01}(\infty) = 1 + \dfrac{1}{b}\left(1 - \sqrt{1 + \dfrac{4b}{3}}\right) \\ \widehat{R}_{14}(\infty) = \dfrac{(3+b)^2}{b^3}\left(1 + \dfrac{2b}{3} - \sqrt{1 + \dfrac{4b}{3}}\right) - \dfrac{2}{b} + \dfrac{1}{3} \\ \widehat{R}_{15}(\infty) = 3\dfrac{3 + 7b + 5b^2 + b^3}{b^3} - \dfrac{9 + 15b + 7b^2}{b^3}\sqrt{1 + \dfrac{4b}{3}} \end{cases}. \tag{10}$$

Using the same approach, we can derive other recurrence relations. Indeed, the equivalent circuit linking nodes 3, 4 and 5, is the same as the equivalent circuit between nodes 0, 1 and 2 when exchanging b and A_m; it is therefore a frame of three equal resistors $3 + b$ in parallel with A_m, hence $B_{m+1} = A_m(3+b)/(b + 3 + A_m)$, which gives the resistance $R_{34}(m+1)$ as a function of $R_{01}(m)$:

$$R_{34}(m+1) = \frac{2}{3}B_{m+1} = \frac{2(3+b)R_{01}(m)}{6 + 2b + 3R_{01}(m)}. \tag{11}$$

Similarly, we construct the globe network iteratively from the $(m-1) \times 3$ tower network, turn it upside down, add one layer and a top node, making the $(m+1) \times 3$ globe network. For example, the 3×3 cylinder network (figure 4(d)) is obtained by adding one layer to the 2×3 cylinder network (figure 4(a)), and $R_{01}(3)$ follows from $R_{01}(2)$ using equation (4). Similarly, the upside down 3×3 tower network (figure 4(e)) is obtained by adding one layer to the upside down 2×3 tower network (figure 4(b)), and $\hat{R}(3)$ follows from $\hat{R}(2)$ using equation (4). When we add one layer plus a top node to the upside down 2×3 tower network (figure 4(b)), then we get the 3×3 globe network (figure 4(f)). As shown above, one layer of b resistors plus one top node is equivalent to one layer of b' resistors. To obtain $\widehat{R}_{12}(m)$, thus, we just use equation (4), replacing b by b', and we have:

$$\widehat{R}_{12}(m+1) = 2b'\frac{2 + \hat{R}(m-1)}{6 + 2b' + 3\hat{R}(m-1)}. \tag{12}$$

Using equation (8), and $b' = 3b/(3+b)$, we can write equation (12) in terms of $\widehat{R}_{12}(m)$ only:

$$\widehat{R}_{12}(m+1) = \frac{4b}{6 + 4b - b\widehat{R}_{12}(m)}. \tag{13}$$

As done above for R_{34} in equation (11), we can also write a recurrence relation for $\widehat{R}_{45}$. Indeed, we just have to replace $R_{01}(m)$ by $\hat{R}$ and b by b':

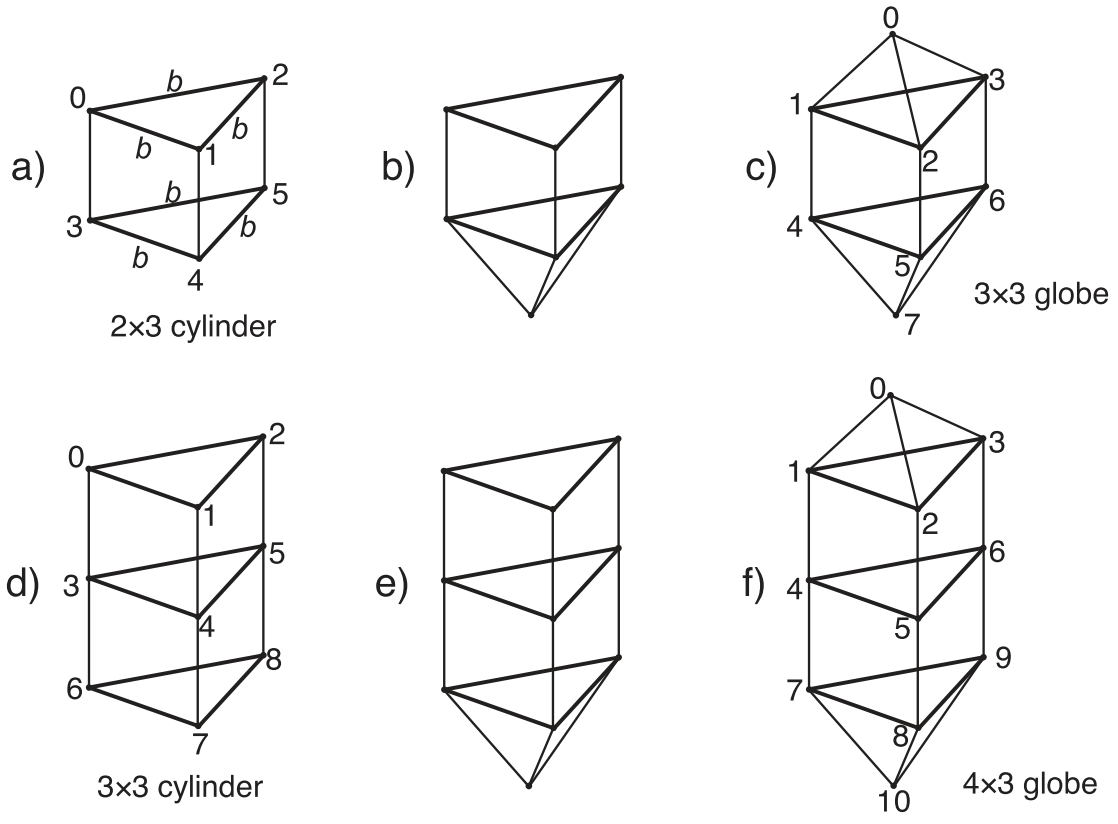

Figure 4. First steps of the iterative construction of $m \times 3$ cylinder and globe networks: (a) a cylinder 2×3 network, (b) after adding a top node and rotating upside down, and (c) adding a top pole, thus obtaining the 3×3 globe network. Similarly: (d) adding a layer to the 2×3 cylinder, thus making the 3×3 cylinder network, (e) after adding a top node to the 2×3 cylinder and rotating upside down, or adding a layer to the rotated tower of figure 4(b), and (f) adding a top pole, thus obtaining the 4×3 globe network.

$$\widehat{R}_{45}(m+1) = \frac{2(3+b')\hat{R}(m-1)}{6+2b'+3\hat{R}(m-1)} = \frac{2(3+2b)\hat{R}(m-1)}{6+4b+(3+b)\hat{R}(m-1)} = \frac{2(3+2b)\widehat{R}_{12}(m)}{6+4b-b\widehat{R}_{12}(m)}. \tag{14}$$

To illustrate the method, we give explicit relations for the first values of m (tables 1 and 2). To start the iteration, we take a frame of three resistors b, which gives $R_{01}(1) = 2b/3$. Adding one pole, using equation (8), we obtain $\hat{R}(1) = 2b(3+b)$; adding a second pole, we obtain $\widehat{R}_{12}(2) = 2b/(3+2b)$. Using recurrence relations in equations (4) and (13), we obtain the explicit expressions for $R_{01}(m)$ and $\widehat{R}_{12}(m)$ given in tables 1 and 2. From equations (1)–(3), we then obtain $R_{03}(m)$, $R_{04}(m)$, $\widehat{R}_{01}(m)$, $\widehat{R}_{14}(m)$ and $\widehat{R}_{15}(m)$.

4. Recurrence relations for closest neighbour internal resistances

Further recurrence relations can be inferred without more calculations. Consider an $m \times 3$ cylinder network and some position inside the network with q layers above and p layers below, with $m = q + p$ (figure 5(a)). The bottom nodes of the upper q layers are $i, i+1$ and $i+2$, with $i = 3(q-1)$, while the top nodes of the lower p layers are $i+3, i+4$ and $i+5$. The equivalent circuit between nodes $i, i+1$ and $i+2$ is a triangle of three resistors $A_q = 3R_{01}(q)/2$. The equivalent circuit between nodes $i+3, i+4$ and $i+5$ is a triangle of three resistors $A_p = 3R_{01}(p)/2$.

Consequently, the whole circuit containing nodes $i, i+1, i+2, i+3, i+4$ and $i+5$ is equivalent to the circuit of figure 5(b), which is identical to the circuit of figure 3(a), replacing b by A_q. Thus, we can use equation (4) and we get:

$$R_{i(i+1)}(m) = 2A_q \frac{2+R_{01}(p)}{6+2A_q+3R_{01}(p)} = R_{01}(q)\frac{2+R_{01}(p)}{2+R_{01}(q)+R_{01}(p)}. \tag{15}$$

The equivalent resistance between nodes $i+3$ and $i+4$ is obtained by exchanging p and q:

$$R_{(i+3)(i+4)}(m) = R_{01}(p)\frac{2+R_{01}(q)}{2+R_{01}(q)+R_{01}(p)}. \tag{16}$$

Table 1. Closest neighbour two-point resistances of the $m \times 3$ cylinder networks for $m \leqslant 5$.

m	1	2	3	4	5
R_{01}	$\frac{2b}{3}$	$\frac{\frac{2b}{3}\frac{3+b}{3+2b}}{3+2b}$	$\frac{\frac{2b}{9}\frac{9+9b+b^2}{(3+b)(1+b)}}{3(3+b)(1+b)}$	$\frac{\frac{2b}{3}\frac{27+45b+18b^2+b^3}{27+54b+30b^2+4b^3}}{(3+2b)(9+12b+2b^2)}$	$\frac{\frac{2b}{3}\frac{81+189b+135b^2+30b^3+b^4}{81+216b+189b^2+60b^3+5b^4}}{3(81+216b+189b^2+60b^3+5b^4)}$
R_{03}		$\frac{1+2b}{3+2b}$	$\frac{3+8b+3b^2}{3(3+b)(1+b)}$	$\frac{9+30b+24b^2+4b^3}{(3+2b)(9+12b+2b^2)}$	$\frac{27+108b+129b^2+52b^3+5b^4}{81+216b+189b^2+60b^3+5b^4}$
R_{04}		$\frac{3+6b+2b^2}{3(3+2b)}$	$\frac{9+24b+15b^2+2b^3}{9(3+b)(1+b)}$	$\frac{27+90b+90b^2+30b^3+2b^4}{3(3+2b)(9+12b+2b^2)}$	$\frac{81+324b+441b^2+246b^3+51b^4+2b^5}{3(81+216b+189b^2+60b^3+5b^4)}$
R_{34}			$\frac{\frac{2b}{9}\frac{3+b}{1+b}}{}$	$\frac{\frac{2b}{3}\frac{(3+b)(9+9b+b^2)}{(3+2b)(9+12b+2b^2)}}{}$	$\frac{2b(3+b)(27+45b+18b^2+b^3)}{3(81+216b+189b^2+60b^3+5b^4)}$
R_{36}				$\frac{3+8b+2b^2}{9+12b+2b^2}$	$\frac{27+108b+123b^2+48b^3+5b^4}{81+216b+189b^2+60b^3+5b^4}$
R_{37}				$\frac{27+90b+84b^2+24b^3+2b^4}{3(3+2b)(9+12b+2b^2)}$	$\frac{81+324b+423b^2+216b^3+39b^4+2b^5}{3(81+216b+189b^2+60b^3+5b^4)}$

Table 2. Closest neighbour two-point resistances of the $m \times 3$ globe networks for $m \leqslant 6$.

m	2	3	4	5	6
$\bar{R}_{12}$	$\frac{2b}{3+2b}$	$\frac{\frac{2b}{3}\frac{3+2b}{(3+b)(1+b)}}{}$	$6b\frac{(3+b)(1+b)}{(3+2b)(9+12b+2b^2)}$	$2b\frac{27+54b+30b^2+4b^3}{(9+15b+5b^2)(9+9b+b^2)}$	$\frac{2b}{3}\frac{(9+15b+5b^2)(9+9b+b^2)}{(1+b)(3+b)(3+2b)(9+12b+b^2)}$
$\bar{R}_{01}$	$\frac{3+4b}{3(3+2b)}$	$\frac{9+18b+7b^2}{9(3+b)(1+b)}$	$\frac{27+72b+54b^2+10b^3}{3(3+2b)(9+12b+2b^2)}$	$\frac{81+270b+297b^2+120b^3+13b^4}{3(9+15b+5b^2)(9+9b+b^2)}$	$\frac{243+972b+1404b^2+882b^3+225b^4+16b^5}{9(1+b)(3+b)(3+2b)(9+12b+b^2)}$
$\bar{R}_{14}$		$\frac{3+7b}{9(1+b)}$	$\frac{27+90b+66b^2+10b^3}{3(3+2b)(9+12b+2b^2)}$	$\frac{81+324b+369b^2+138b^3+13b^4}{3(9+15b+5b^2)(9+9b+b^2)}$	$\frac{81+351b+459b^2+201b^3+16b^4}{9(1+b)(3+b)(3+2b)(9+12b+b^2)}$
$\bar{R}_{15}$		$\frac{9+24b+13b^2}{9(3+b)(1+b)}$	$\frac{27+90b+84b^2+22b^3}{3(3+2b)(9+12b+2b^2)}$	$\frac{81+324b+423b^2+210b^3+31b^4}{3(9+15b+5b^2)(9+9b+b^2)}$	$\frac{243+1134b+1890b^2+1386b^3+429b^4+40b^5}{9(1+b)(3+b)(3+2b)(9+12b+b^2)}$
$\bar{R}_{45}$			$2b\frac{3+2b}{9+12b+2b^2}$	$6b\frac{(3+2b)(3+4b+b^2)}{(9+15b+5b^2)(9+9b+b^2)}$	$\frac{27+54b+30b^2+4b^3}{9(1+b)(3+b)(9+12b+b^2)}$
$\bar{R}_{47}$				$\frac{9+27b+13b^2}{3(9+15b+5b^2)}$	$\frac{243+1134b+1728b^2+1080b^3+231b^4+16b^5}{9(1+b)(3+b)(3+2b)(9+12b+b^2)}$
$\bar{R}_{48}$				$\frac{81+324b+423b^2+216b^3+37b^4}{3(9+15b+5b^2)(9+9b+b^2)}$	$\frac{243+1134b+1890b^2+1404b^3+459b^4+52b^5}{9(1+b)(3+b)(3+2b)(9+12b+b^2)}$

Similarly, we can apply equation (2) replacing b by A_q, and we get:

$$\begin{cases} R_{i(i+3)}(m) = \dfrac{3R_{i(i+1)}(m) - A_q(2 - A_q)}{{A_q}^2} = \dfrac{2 + 3(R_{01}(p) + R_{01}(q))}{3(2 + R_{01}(p) + R_{01}(q))} \\ R_{i(i+4)}(m) = \dfrac{(3 + 3A_q + {A_q}^2)R_{i(i+1)}(m) - A_q(2 + A_q)}{{A_q}^2} = \dfrac{2 + 3(R_{01}(p) + R_{01}(q) + R_{01}(p)R_{01}(q))}{3(2 + R_{01}(p) + R_{01}(q))} \end{cases}. \tag{17}$$

These expressions, as expected, are symmetrical in p and q, while equations (15) and (16) are symmetrical between each other. As an example of application, we give in table 1 the explicit expressions obtained using equation (17) for $R_{36}(4)$, $R_{37}(4)$, $R_{36}(5)$ and $R_{37}(5)$.

When considering these internal resistances for a large network, we can use the obtained limit equation (6) for R_{01}. When both p and q are sufficiently large, we have $R_{01}(p) = R_{01}(q) = R_{01}(\infty)$, which, when inserted into equations (15) and (17), gives:

$$\begin{cases} R_{i(i+1)}(\infty) = \dfrac{R_{01}(\infty)(2 + R_{01}(\infty))}{2(1 + R_{01}(\infty))} = \dfrac{2b}{3\sqrt{1 + \frac{4b}{3}}} \\ R_{i(i+3)}(\infty) = \dfrac{1 + 3R_{01}(\infty)}{3(1 + R_{01}(\infty))} = 1 - \dfrac{2}{3\sqrt{1 + \frac{4b}{3}}} \\ R_{i(i+4)}(\infty) = \dfrac{2 + 3R_{01}(\infty)(2 + R_{01}(\infty))}{6(1 + R_{01}(\infty))} = \dfrac{1 + 2b}{3\sqrt{1 + \frac{4b}{3}}} \end{cases}. \tag{18}$$

The same approach can be used for the internal resistances of an $m \times 3$ globe network. It has $m - 1$ horizontal frames. When separating the top $m_1 - 1$ horizontal frames, then $m_2 - 1$ frames remain below, thus $m = m_1 + m_2 - 1$. The lowest indices in the upper group of frames are i, $i + 1$ and $i + 2$ with $i = 3(m_1 - 1) - 2$. The separation is thus characterized by:

$$\begin{cases} i = 3m_1 - 5 \\ j = 3m_1 - 2 \\ m = m_1 + m_2 - 1 \end{cases}. \tag{19}$$

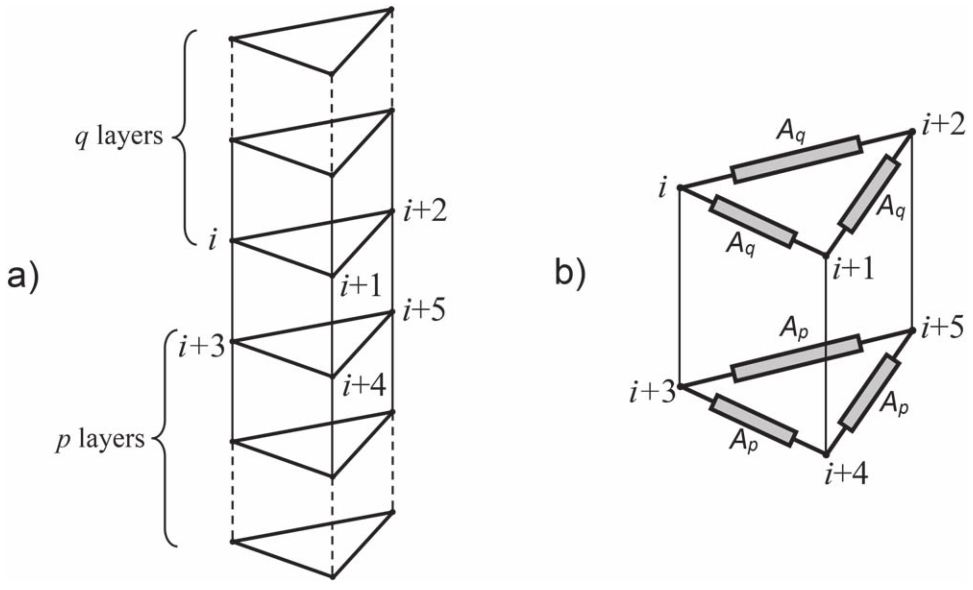

Figure 5. (a) Inner part of an $m \times 3$ cylinder or globe network. (b) Equivalent diagram of the network made by the internal nodes i, $i+1, i+2, i+3, i+4$ and $i+5$.

To find the internal resistances, we apply equations (15)–(17), replacing $R_{01}(q)$ and $R_{01}(p)$, by $\hat{R}_{01}(m_1 - 1)$ and $\hat{R}_{01}(m_2 - 1)$, respectively. Using equation (8), we obtain:

$$\begin{cases} \widehat{R}_{i(i+1)}(m) = \dfrac{4\widehat{R}_{01}(m_1)}{4 - \widehat{R}_{01}(m_1)\widehat{R}_{01}(m_2)} \\ \widehat{R}_{(i+3)(i+4)}(m) = \dfrac{4\widehat{R}_{01}(m_2)}{4 - \widehat{R}_{01}(m_1)\widehat{R}_{01}(m_2)} \\ \widehat{R}_{i(i+3)}(m) = \dfrac{4 + 4\widehat{R}_{01}(m_1) + 4\widehat{R}_{01}(m_2) - 5\widehat{R}_{01}(m_1)\widehat{R}_{01}(m_2)}{3(4 - \widehat{R}_{01}(m_1)\widehat{R}_{01}(m_2))} \\ \widehat{R}_{i(i+4)}(m) = \dfrac{4 + 4\widehat{R}_{01}(m_1) + 4\widehat{R}_{01}(m_2) + \widehat{R}_{01}(m_1)\widehat{R}_{01}(m_2)}{3(4 - \widehat{R}_{01}(m_1)\widehat{R}_{01}(m_2))} \end{cases}. \tag{20}$$

In table 2, we give the explicit expressions obtained using equation (20) for $\widehat{R}_{47}(5)$, $\widehat{R}_{48}(5)$, $\widehat{R}_{47}(6)$ and $\widehat{R}_{48}(6)$. As another application, suppose we are interested in the expression of the middle internal resistance between nodes 13 and 16 of the 11×3 globe network. Using equation (19), we have $m_1 = 6 = m_2$. We then take $\widehat{R}_{01}(6)$ from table 2 and we get, directly from equation (20):

$$\widehat{R}_{13-16}(11) = \frac{243 + 1215b + 2052b^2 + 1449b^3 + 405b^4 + 31b^5}{3(243 + 891b + 1188b^2 + 693b^3 + 165b^4 + 11b^5)}. \tag{21}$$

Now we know how to obtain the expressions for the resistance between all closest-neighbour nodes in $m \times 3$ cylinder and globe networks, which all result from the basic recurrence relation of equation (4). When the nodes are more distant, some relations can also be inferred.

5. Recurrence relations for two-point resistances across one frame

Following the method used above, we start from a fundamental network (figure 6(a)), obtained by adding an intermediate frame to the network of figure 5(b). To account for possible networks above and below, we introduce in figure 6(a) resistances A and B, similar to A_q and A_p, respectively, in figure 5(b). While most equivalent resistances are already known from the previous sections, we are interested in the equivalent resistance between nodes 0 and 6, and in the equivalent resistance between nodes 0 and 7. The previous recurrence relations cannot be applied, neither the Y-Δ transform, so we use van Steenwijk's method.

To apply this method, consider the configuration of currents when a unity current is injected at node 0 and current 1/8 is extracted from the other eight nodes (figure 6(a)). By symmetry, the currents between nodes 1 and 2, between nodes 4 and 5 and between nodes 7 and 8 are zero. Designating by α and β the currents from node 0 to

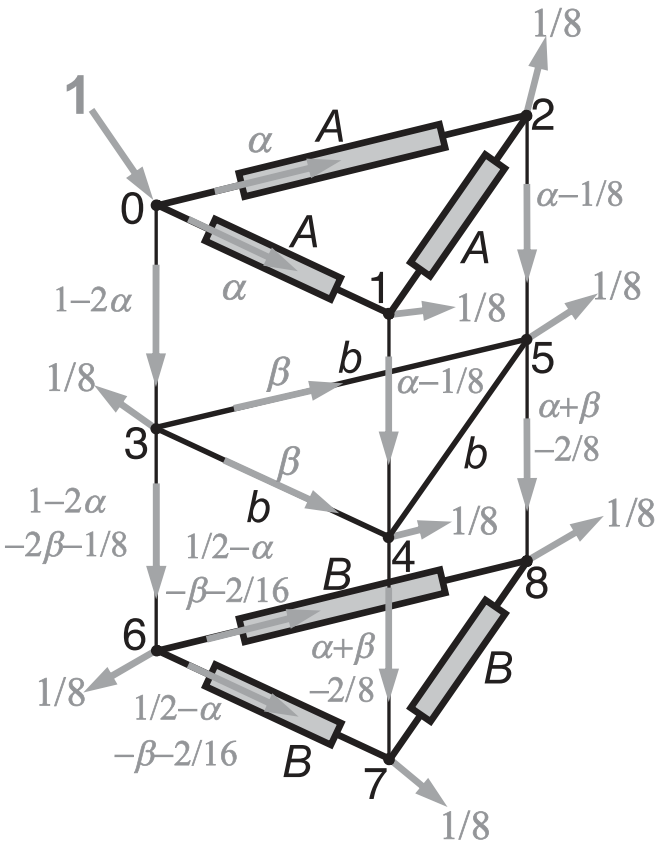

Figure 6. Equivalent diagram of an internal part of an $m \times 3$ cylinder or globe network, displaying nine nodes and one intermediate frame layer.

node 1, and from node 3 to node 4, respectively, all other currents are fixed by symmetry or current conservation (figure 6(a)). The currents α and β are constrained by closed loops 3-4-1-0 and 6-7-4-3, giving:

$$\begin{cases} b\beta - \left(\alpha - \frac{1}{8}\right) - A\alpha + (1 - 2\alpha) = 0 \\ B\left(\frac{1}{2} - \alpha - \beta - \frac{1}{8}\right) - \left(\alpha + \beta - \frac{2}{8}\right) - b\beta + \left(\frac{7}{8} - 2\alpha - 2\beta\right) = 0 \end{cases}, \tag{22}$$

or:

$$\begin{cases} b\beta = -\frac{9}{8} + (3 + A)\alpha \\ (3 + B)\alpha + (3 + b + B)\beta = \frac{9 + 3B}{8} \end{cases}, \tag{23}$$

which gives:

$$\begin{cases} \alpha = \frac{3}{8}\dfrac{(3 + B)b + 3(3 + b + B)}{(3 + B)b + (3 + A)(3 + b + B)} \\ \beta = \frac{3}{8}\dfrac{A(3 + B)}{(3 + B)b + (3 + A)(3 + b + B)} \end{cases}. \tag{24}$$

The total voltage drop V_{06} from node 0 to node 6 is then:

$$V_{06} = \frac{15}{8} - 4\alpha - 2\beta = \frac{3}{8}\frac{9 + 6b + 9A + (3 + b)B + 3AB + 5Ab}{(3 + B)b + (3 + A)(3 + b + B)}. \tag{25}$$

To obtain R_{06}, van Steenwijk's method works as follows [6–8]. We subtract from the current configuration shown in figure 6(a), the current configuration obtained when injecting a unity current at node 6 and extracting current $1/8$ at all other nodes. Then, we obtain a total current configuration with current $1 + 1/8 = 9/8$ injected at node 0 and extracted at node 6. The resistance is then the total voltage drop divided by the current $9/8$. Here, it is not needed to calculate the second current configuration when injecting a unity current at node 6.

This is indeed the configuration depicted in figure 6(a), exchanging A and B. The potential drop $\tilde{V}_{06}$ from 0 to 6 in the second configuration is thus given by equation (25), exchanging A and B. In the following, in general, we use the convention that, for any quantity $F(A, B)$, then $\tilde{F}(A, B) = F(B, A)$. We then obtain:

$$R_{06} = \frac{8}{9}(V_{06} + \tilde{V}_{06}) = 2\frac{3 + 2b + (2 + b)(A + B) + AB}{3(3 + 2b) + (3 + b)(A + B) + AB}. \tag{26}$$

Similarly, we have:

$$R_{07} = \frac{8}{9}(V_{07} + \tilde{V}_{07}), \tag{27}$$

with:

$$V_{07} = \frac{15}{8} - 4\alpha - 2\beta + B\left(\frac{1}{2} - \alpha - \beta - \frac{2}{16}\right) = 3\frac{5 + B}{8} - (4 + B)\alpha - (2 + B)\beta, \tag{28}$$

and we obtain:

$$R_{07} = \frac{2}{3}\frac{3(3 + 2b) + 3(2 + b)(A + B) + (3 + b)AB}{3(3 + 2b) + (3 + b)(A + B) + AB}. \tag{29}$$

In equations (26) and (29), the terms are rearranged to separate the symmetric combinations $A + B$ and AB. In the simplest case, we have $A = B = b$, giving:

$$\begin{cases} R_{06}(3) = 2\dfrac{1 + b}{3 + b} \\ R_{07}(3) = \dfrac{2}{9}\dfrac{9 + 18b + 9b^2 + b^3}{(1 + b)(3 + b)} \end{cases}. \tag{30}$$

In the case of an extended cylinder network, we have, as in the previous section, $A = 3R_{01}(q)/2$ and $B = 3R_{01}(q)/2$, thus:

$$\begin{cases} R_{i(i+6)}(m) = \dfrac{2}{3}\dfrac{4(3 + 2b) + 6(2 + b)(R_{01}(p) + R_{01}(q)) + 9R_{01}(p)R_{01}(q)}{4(3 + 2b) + 2(3 + b)(R_{01}(p) + R_{01}(q)) + 3R_{01}(p)R_{01}(q)} \\ R_{i(i+7)}(m) = \dfrac{2}{3}\dfrac{4(3 + 2b) + 6(2 + b)(R_{01}(p) + R_{01}(q)) + 3(3 + b)R_{01}(p)R_{01}(q)}{4(3 + 2b) + 2(3 + b)(R_{01}(p) + R_{01}(q)) + 3R_{01}(p)R_{01}(q)} \end{cases}. \tag{31}$$

As an example of application, consider $R_{39}(5)$. We just apply equation (31) with $p = q = 2$ and the expression of $R_{01}(2)$ from table 1. We obtain:

$$R_{39}(5) = 2\frac{27 + 90b + 99b^2 + 40b^3 + 5b^4}{81 + 216b + 189b^2 + 60b^3 + 5b^4}. \tag{32}$$

From equation (31), we also obtain, for example, the situation of large networks above and below, using equation (6):

$$R_{i(i+6)}(\infty) = \frac{2}{3}\frac{4(3 + 2b) + 12(2 + b)R_{01}(\infty) + 9R_{01}(\infty)^2}{4(3 + 2b) + 4(3 + b)R_{01}(\infty) + 3R_{01}(\infty)^2} = \frac{2}{b}\left(1 + b - \sqrt{1 + \frac{4b}{3}}\right). \tag{33}$$

We can also derive resistances in $m \times 3$ globe networks. To get the expression of $\widehat{R}_{17}(4)$, for example, we apply equation (29) with $A = B = b'$ and we get:

$$\widehat{R}_{17}(4) = \frac{2}{3}\frac{3 + 4b}{3 + 2b}. \tag{34}$$

We can also obtain the expression of the internal resistance across one frame in the $m \times 3$ globe network. As in section 4 above, we replace in equation (31) $R_{01}(q)$ and $R_{01}(p)$, by $\hat{R}_{01}(m_1 - 1)$ and $\hat{R}_{01}(m_2 - 1)$, respectively, and then, using equation (8), we obtain:

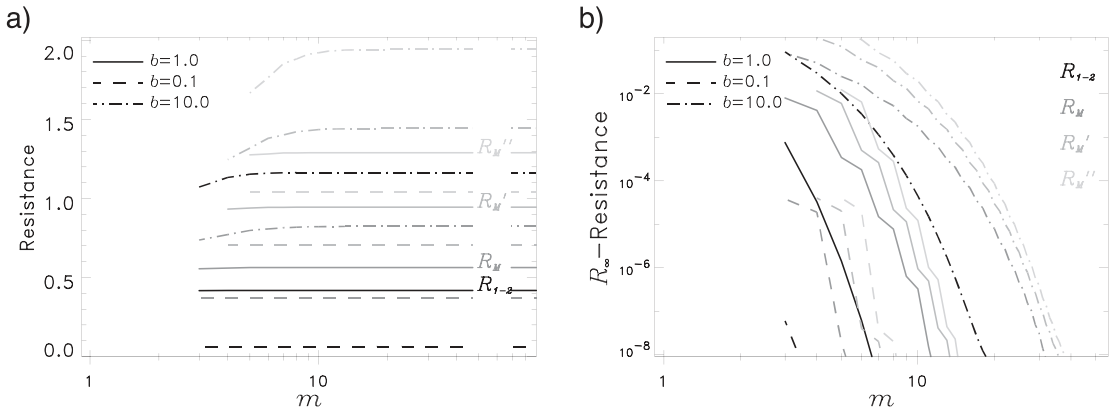

Figure 7. Convergence of two-point resistances in the $m \times 3$ globe network as a function of m. (a) Edge resistance (R_{12}), resistance between two middle internal nodes along the axis (R_M), and resistance between two internal nodes along the axis separated by one middle node (R_M'). (b) Difference between the limit value at large m (R_∞) and the resistance for a given m. Three values of b are considered: $b = 1$ (full line), $b = 0.1$ (dotted line) and $b = 10$ (dash-dot line).

$$\widehat{R}_{i(i+6)}(m) = \frac{2}{3}\frac{4(3+2b)+2(3+b)(\widehat{R}_{01}(m_1)+\widehat{R}_{01}(m_2))-4b\widehat{R}_{01}(m_1)\widehat{R}_{01}(m_2)}{4(3+2b)-2b(\widehat{R}_{01}(m_1)+\widehat{R}_{01}(m_2))}. \tag{35}$$

This method can be generalized to obtain two-point resistances across two and more intermediate frames (appendix B).

6. Discussion

From the recurrence relations in equations (4), (11), (13)–(14), (15)–(17) and (20), the exact equations for the limits for large numbers of layers are obtained as a function of the resistance b. When m is increased, the resistance between nodes of the $m \times 3$ globe network increases to approach the limit value (figure 7(a)). The difference between the limit and the actual value for a given m and a given choice of b, however, is not necessarily negligible, even for values of m larger than 10 (figure 7(b)). Indeed, this difference, for a given value of m, is larger for larger values of b. In addition, it is larger for the central internal resistance along the axis (R_M) than for the edge resistance R_{12} and even larger for the central internal resistance along the axis with one intermediate frame (R_M'). The equations for the limit values, therefore, can be used instead of the actual values as a function of m, but with some caution depending on the considered domain of m, on the required precision, and depending on the particular pair of nodes considered.

Our explicit solutions are useful to study practical cases. By replacing the basic resistors by reactive components (capacitors and inductances), the impedance response can also be calculated analytically. For example, equations (30) and (34) can be used to evaluate the effect of adding covering resistances at the top and at the bottom of a three-layer triangular pile of a basic reactive element (figure 8). As an illustration, we take a basic element made of an inductance L_0 in series with a resistor R_0 and a capacitance C_0 in parallel (figure 8(a)). Resistance b in our model is replaced by a complex impedance Z_0 depending on frequency f [30]:

$$Z_0(0) = i\omega L_0 + \frac{R_0}{1 + i\omega R_0 C_0}, \tag{36}$$

where i is here the square root of -1 and $\omega = 2\pi f$. The variations with frequency of the impedances $Z_{06}(3)$ and $\widehat{Z}_{17}(4)$ (figure 8(b)) show a transition from a low-frequency limit, obtained when b is replaced by R_0, to the high-frequency limits 2 from equation (30) and 4/3 from equation (34), respectively. The variation with frequency during the transition depends on the R_0, L_0 and C_0 parameter values. The effect of piling up more layers on the frequency response can easily be studied as the analytical expressions can be obtained for any number of layers.

The expressions we have obtained above were first checked using a numerical code based on node reduction using Kennelly's theorem. These expressions can also be compared with the exact solutions of the $m \times 3$ cylinder and globe networks given by Tan's RT method [29]. The obtained expressions of two-point resistances with m smaller than 4, elaborated in detail in appendix C, correspond exactly to the explicit expressions given in tables 1 and 2. As the two methods are completely independent, the comparison is highly meaningful. In Tan's RT method, general and elegant expressions are obtained, and particular simple cases are worked out progressively in a recursive manner. Here, an opposite approach is used. We start from a simple fundamental

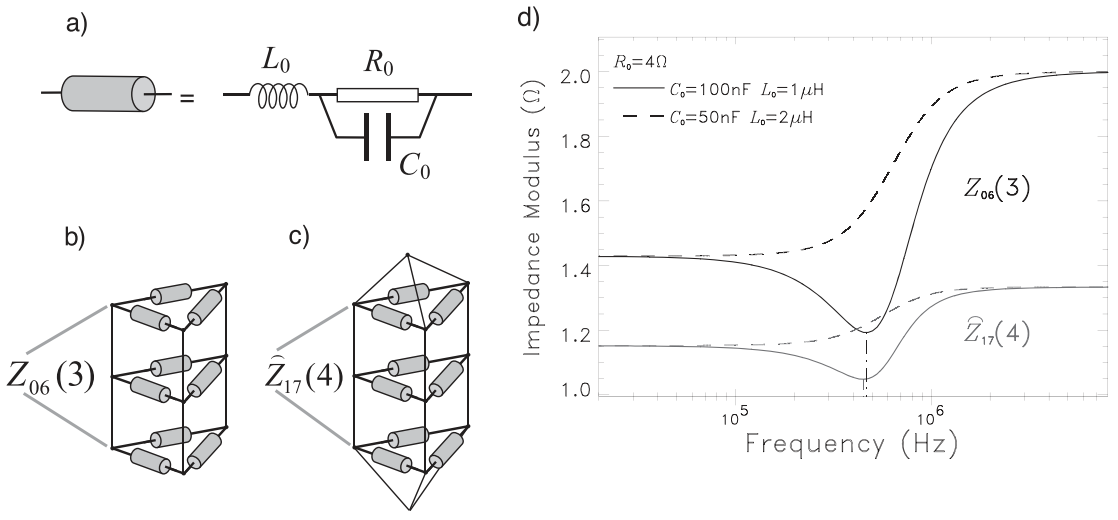

Figure 8. Comparison of the impedance response. (a) Resistor b of the horizontal frame is replaced by an element including an inductance L_0 and a resistance R_0 in parallel with a capacitance C_0. (b) Considered impedance between top and bottom frame of the 3×3 cylinder network. (c) Considered impedance between top and bottom frame of the 4×3 globe network. (d) Modulus of impedance as a function of frequency for two different values of L_0 and C_0, keeping $R_0 = 2\,\Omega$.

unit cell (figure 3(a)), then more and more complicated networks are built up by combining different variants of the unit cell.

Beyond the actual practical cases considered in this paper, the method described here appears useful to complement the derivations using general methods such as the RT method or Green functions [19–22]. Our method can be analyzed as follows. First, we have stated the relations between two-point resistances resulting from the general symmetry properties, in particular near the terminal nodes. Then, we have established and solved a root network (figure 3), from where various types of iterative relations follow. In this case, we have obtained all two-point resistances between closest neighbours. Finally, we have shown that other basic root networks can be solved (figure 6), here with one or two intermediate frames (appendix B), and further relations have been inferred. This approach is general and should be attempted in other types of networks, such as the $m \times 4$ cylinder and globe networks, for which explicit expressions have been also obtained from the RT method [29]. In more complex networks with symmetries of higher orders, even if it turns out that our direct approach is not simpler than the general RT summations and that not all two-point resistances potentially can be derived, even partial recurrence relations will be useful to obtain limit cases and discuss the relevance of a given network to model a situation of concern. Analytical expressions and limit values are meaningful tools to evaluate the capacity of a given network topology to account for observations. Our approach does not replace more powerful and complete calculation methods, but it provides a useful assessment of properties.

7. Conclusions

In this paper, we have shown how the explicit expressions for all closest neighbour resistances in $m \times 3$ scaffolding (cylinder) and globe networks result from a single recurrence relation $R_{01}(m+1) = 2b(2 + R_{01}(m))/(6 + 2b + 3R_{01}(m))$, equation (4), obtained by using only the Y-Δ transform (Kennelly's theorem). Furthermore, by using van Steenwijk's method, we have obtained explicit expressions for resistances across one (section 5), two or more intermediate frames (appendix B). These calculations illustrate how the symmetry properties (planar and 3-fold invariance around a central axis) can be combined with a repeated pattern of resistors.

Our expressions coincide with the explicit results that can be inferred from the general expressions given by the RT method [29]. Thus, our derivation provides a completely independent verification, and this comparison enhances the confidence we can have in both methods to obtain explicit analytical expressions. Such analytical results are useful to check numerical codes in practical applications, and to evaluate the relevance of a given network topology to model the physical system of concern and compare with experimental data.

In the case of the $m \times 3$ networks considered here, our analysis complements the general results by giving an appreciation of the consequences of the repetition of basic unit cells to construct extended networks with 3-fold invariance. By such analysis, we learn to be aware of the different types of relations of the two-point resistance appearing in such three-dimensional networks and we understand their origin. In complex networks, it will

remain useful to complement general expressions by relations which are direct consequences of the rotational and translational symmetry properties [33]. Since networks offer a modelling approach with a rapidly growing range of applications, from material science [10] and geophysics [2, 14–16] to biological systems [34], the exploration of the rich properties of resistor networks in three dimensions and their mathematical descriptions will remain a vast research area for the coming years.

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Data availability statement

No new data were created or analysed in this study.

Appendix A

Two-plus-one legs corner theorem

Consider a node, labelled 0, with three branches connected to a network with a plane of symmetry containing this node and one of the branches (figure A1). Without loss of generality, we assume that the resistance of the branch in the symmetry plane, connecting node 0 to a node labelled 3, is unity and the resistance connecting 0 to the two other nodes 1 and 2 is b. The equivalent diagram between nodes 1, 2 and 3, capturing the rest of the network, is an isosceles triangle with one equivalent resistance A between nodes 1 and 2, and two equal resistances B for the branches 3–1 and 3–2 (figure A2). This arrangement constitutes the reduced network in the considered problem.

Below we show that the two-point resistances between the nodes are related, and we will derive expressions with resistance b as a parameter. This demonstration is more general than the case treated previously, which was assuming $b = 1$ [8]. To express the two-point resistances, we use van Steenwijk's method. For this purpose, we derive the current distribution corresponding to the injection of a unity current, and extracting current $1/3$ from each of the other three nodes, for each class of possible injection nodes of the problem. In the present case, three configurations need to be considered (figure A2).

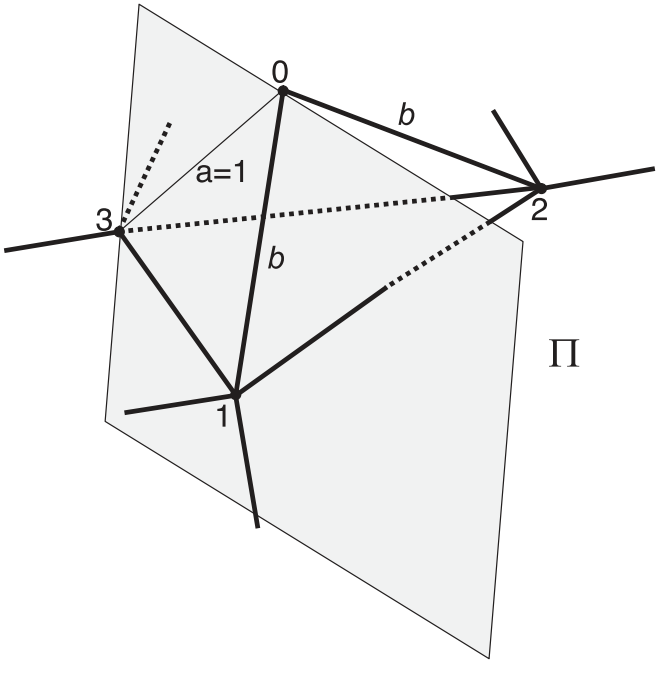

Figure A1. General case of a corner node 0 with three legs in a network with a plane of symmetry Π containing node 0 and one leg.

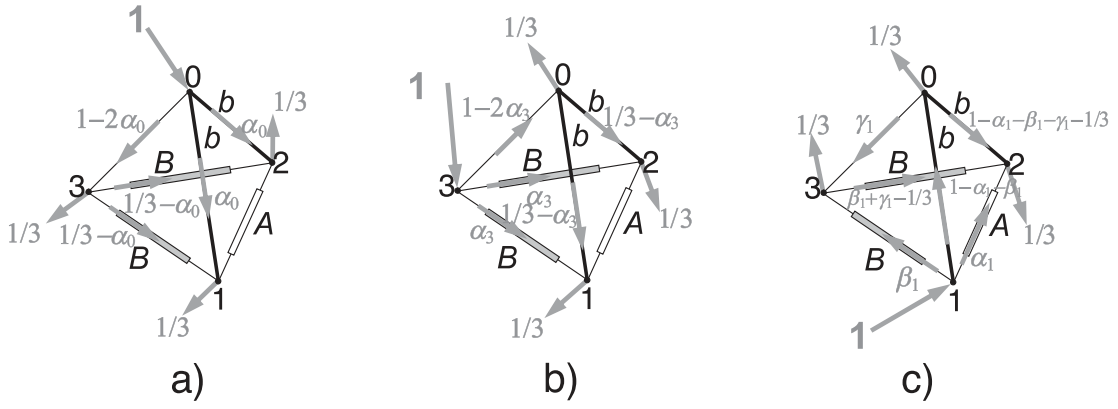

Figure A2. Equivalent diagram in the situation of figure A1. Node 0 is the corner node and node 3 is in the plane of symmetry with node 0. Resistance of branch 0–3 is 1, and b for branches 0–1 and 0–2. Equivalent resistance between nodes 1 and 2 is A, and B between nodes 3 and 1, and between nodes 3 and 2. (a) Current configuration in the equivalent network when a unity current is injected at node 0. (b) Current configuration in the equivalent network when a unity current is injected at node 3. (c) Current configuration in the equivalent network when a unity current is injected at node 1. Current is counted positive in the direction of the arrow.

In the first configuration (figure A2(a)), unity current is injected at node 0. Calling α_0 the current in branch 0–2, all other currents in the reduced network can be inferred by symmetry or by current conservation. The current between nodes 1 and 2 is zero. Using the closed loop 0–1–3, we have:

$$b\alpha_0 = 1 - 2\alpha_0 + B\left(\frac{1}{3} - \alpha_0\right), \tag{A1}$$

giving:

$$\alpha_0 = \frac{1}{3}\frac{3 + B}{2 + b + B}. \tag{A2}$$

In the second configuration (figure A2(b)), unity current is injected at node 3. Calling α_3 the current in branch 3–1, similarly, all other currents in the reduced network can be inferred by symmetry or by current conservation, with zero again between nodes 1 and 2. Since the configuration of figure A2(b) is the same as the configuration of figure A2(a) when b and B are exchanged, we have:

$$\alpha_3 = \frac{1}{3}\frac{3 + b}{2 + b + B}. \tag{A3}$$

In the third configuration (figure A2(c)), when unity current is injected at node 1, the planar symmetry of the network is not verified by the currents and we need to introduce three unknown currents: α_1 (between nodes 1 and 2), β_1 (between nodes 1 and 3) and γ_1 (between nodes 0 and 3). These unknowns are constrained using the closed loops 1–2–3, 1–3–0 and 1–2–0 giving, respectively:

$$\begin{cases} A\alpha_1 = B\left(2\beta_1 + \gamma_1 - \dfrac{1}{3}\right) \\ B\beta_1 = \gamma_1 + b(1 - \alpha_1 - \beta_1) \\ A\alpha_1 = b\left(2 - 2\alpha_1 - 2\beta_1 - \gamma_1 - \dfrac{1}{3}\right) \end{cases}. \tag{A4}$$

From equation (A4), eliminating γ_1, which is not needed afterwards, we extract the values of α_1 and β_1:

$$\begin{cases} \alpha_1 = \dfrac{4}{3}\dfrac{Bb}{AB + Ab + 2Bb} \\ \beta_1 = \dfrac{AB + 3ABb + 5Ab + 3Ab^2 + 2Bb + 2Bb^2}{3(2 + b + B)(AB + Ab + 2Bb)} \end{cases}. \tag{A5}$$

To obtain the two-point resistance R_{ij} between nodes i and j, we subtract the voltage drop resulting from the current configuration with injection at node i, from the voltage drop resulting from the current configuration with injection at node j, which gives the total voltage drop corresponding to the total current $1 + 1/3$ injected at node i and extracted at node j. Thus, we have:

$$\begin{cases} R_{03} = \dfrac{3}{4}(1 - 2\alpha_0 + 1 - 2\alpha_3) = \dfrac{b + B}{2 + b + B} \\ R_{12} = \dfrac{3}{4}2A\alpha_1 = \dfrac{2ABb}{AB + Ab + 2Bb} \\ R_{01} = \dfrac{3}{4}b(\alpha_0 + 1 - \alpha_1 - \beta_1) = b\dfrac{2AB + AB^2 + Ab + ABb + 2Bb + B^2b}{(2 + b + B)(AB + Ab + 2Bb)} \\ R_{13} = \dfrac{3}{4}B(\alpha_3 + \beta_1) = B\dfrac{AB + ABb + 2Ab + Ab^2 + 2Bb + Bb^2}{(2 + b + B)(AB + Ab + 2Bb)} \end{cases}. \tag{A6}$$

The expression for R_{03} corresponds to the unity resistor in parallel with $b + B$, as expected. From the first and second equations in equation (A6), we extract A and B:

$$\begin{cases} B = \dfrac{(2 + b)R_{03} - b}{1 - R_{03}} \\ A = \dfrac{2BbR_{12}}{2Bb - (b + B)R_{12}} \end{cases}, \tag{A7}$$

which we can inject into the third and fourth equations, and we can express R_{01} and R_{13} in terms of R_{03}, R_{12} and b:

$$\begin{cases} 4R_{01} = R_{12} + b^2R_{03} + b(2 - b) \\ 4R_{13} = R_{12} + (2 + b)^2R_{03} - b(2 + b) \end{cases}. \tag{A8}$$

These relations can be applied to any terminal node of the $m \times n$ cylinder network, for example node 0. Node 2 and node 3 of figure A1 are then node $n - 1$ and node n, respectively, of the cylinder network. We then have:

$$\begin{cases} 4R_{01} = R_{1(n-1)} + b^2R_{0n} + b(2 - b) \\ 4R_{1n} = R_{1(n-1)} + (2 + b)^2R_{0n} - b(2 + b) \end{cases}. \tag{A9}$$

However, by rotational invariance, we have, in the $m \times n$ cylinder network, $R_{1(n-1)} = R_{12}$ and $R_{1n} = R_{0(n+1)}$, and we can rewrite equation (A9) as:

$$\begin{cases} 4R_{01} = R_{12} + b^2R_{0n} + b(2 - b) \\ 4R_{0(n+1)} = R_{12} + (2 + b)^2R_{0n} - b(2 + b) \end{cases}. \tag{A10}$$

In the case of the $m \times 3$ cylinder network, we have the additional condition that $R_{12} = R_{01}$, and equation (A10) leads to:

$$\begin{cases} 3R_{01} = b^2R_{03} + b(2 - b) \\ 4R_{04} = R_{01} + (2 + b)^2R_{03} - b(2 + b) \end{cases}. \tag{A11}$$

Expressing R_{03} as a function of R_{01} in the second equation and eliminating R_{03} from the second equation, we get equation (2), which is the expression used to obtain the results of table 1.

Appendix B

Two-point resistances across two and more intermediate frames

The method presented in section 5 can be generalized to more intermediate levels. Here, we illustrate the case of two frames, shown in figure B1. Consider the configuration of currents when a unity current is injected at node 0 and current 1/11 extracted from the other eleven nodes (figure B1). By symmetry, the currents between nodes 1 and 2, between nodes 4 and 5, between nodes 7 and 8, and between nodes 10 and 11, are zero. We follow the scheme of figure 6 and introduce three unknown currents α, β and γ from node 0 to node 1, from node 3 to node 4 and from node 6 to node 7, respectively; all other currents are fixed by symmetry or current conservation (figure B1). The three unknown currents are constrained by the closed loops 3-4-1-0, 6-7-4-3 and 9-10-7-6, giving:

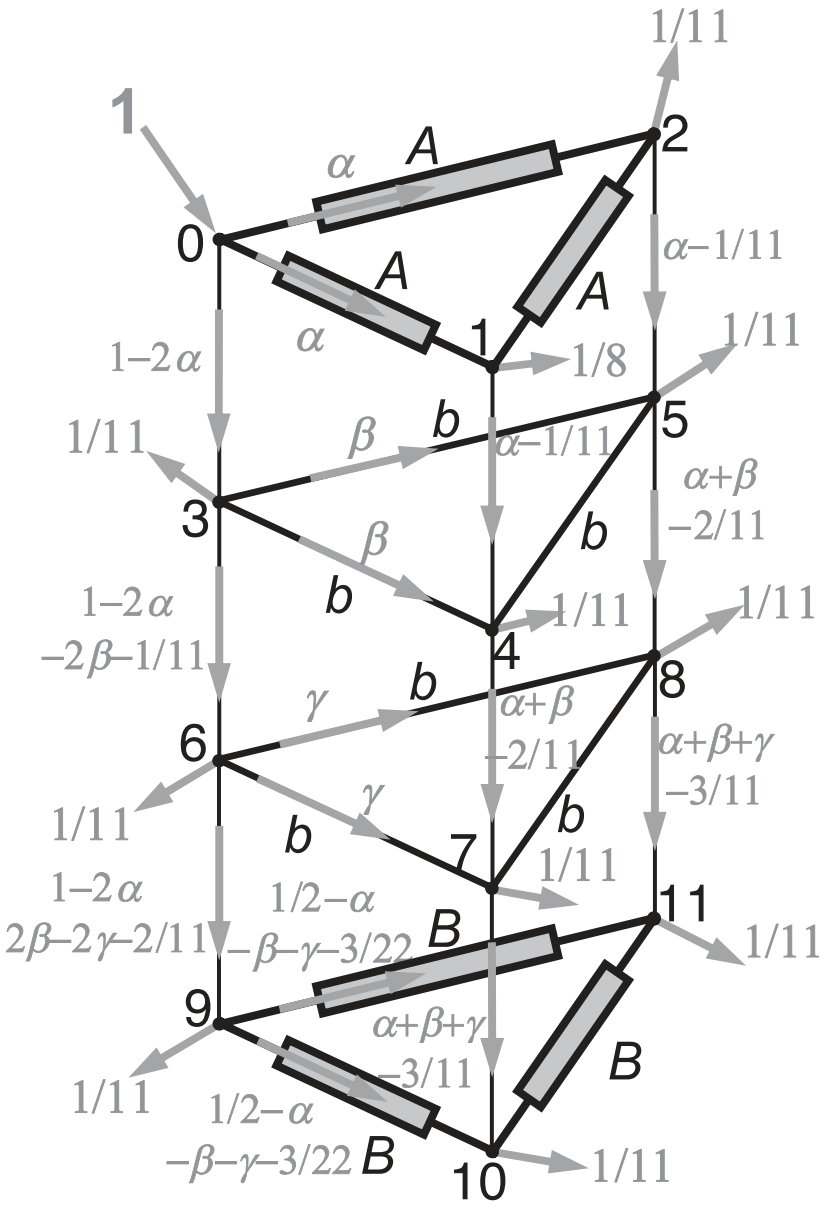

Figure B1. Equivalent diagram of an internal part of an $m \times 3$ cylinder or globe network, displaying 12 internal nodes and two intermediate frame layers.

$$\begin{cases} b\beta - \left(\alpha - \frac{1}{11}\right) - A\alpha + (1 - 2\alpha) = 0 \\ b\gamma - \left(\alpha + \beta - \frac{2}{11}\right) - b\beta + \left(1 - 2\alpha - 2\beta - \frac{1}{11}\right) = 0 \\ B\left(\frac{1}{2} - \alpha - \beta - \gamma - \frac{3}{22}\right) - \left(\alpha + \beta + \gamma - \frac{3}{11}\right) - b\gamma + \left(\frac{1}{11} - 2\alpha - 2\beta - 2\gamma - \frac{2}{11}\right) = 0 \end{cases}, \quad \text{(B1)}$$

hence:

$$\begin{cases} b\beta = -\frac{12}{11} + (3 + A)\alpha \\ b\gamma = -\frac{12}{11} + 3\alpha + (3 + b)\beta \\ (3 + B)\alpha + (3 + B)\beta + (3 + b + b)\gamma = 4\frac{3 + B}{11} \end{cases}. \quad \text{(B2)}$$

Now, before solving explicitly, we establish an intermediate general relation between resistance R_{09} and the edge resistances R_{01} and R_{9-10}. The total voltage drop V_{09} from node 0 to node 9 is:

$$V_{09} = \frac{30}{11} - 6\alpha - 4\beta - 2\gamma. \tag{B3}$$

Taking β and γ from equation (B2), we can express V_{09} in terms of α only:

$$V_{09} = \frac{30}{11} - 6\alpha - 4\beta - 2\gamma = \frac{6}{11b^2}[12 + 16b + 5b^2 - (1 + b)(3 + b + A)11\alpha]. \tag{B4}$$

To express resistance R_{09} using the current configuration of figure B1, we subtract, from the current configuration of figure B1, the current configuration obtained when injecting a unity current at node 9 and extracting current $1/11$ at all other nodes. Then, we obtain a total current configuration with current $1 + 1/11 = 12/11$ injected at node 0 and extracted at node 9. The resistance is now the total voltage drop divided by the current $12/11$. As in section 5, it is not necessary to calculate the second current configuration when injecting a unity current at node 9. This is indeed the configuration depicted in figure B1, exchanging A and B. We thus obtain an expression for R_{09}:

$$R_{09} = \frac{11}{12}(V_{09} + \tilde{V}_{09}) = \frac{1}{2b^2}[2(12 + 16b + 5b^2) - (1 + b)(3 + b + A)11\alpha - (1 + b)(3 + b + B)11\tilde{\alpha}]. \tag{B5}$$

However, we also have:

$$\begin{cases} R_{01} = \dfrac{11}{12}2A\alpha \\ R_{9-10} = \dfrac{11}{12}2B\tilde{\alpha} \end{cases}. \tag{B6}$$

Consequently:

$$R_{09} = \frac{1}{b^2}\left[(2 + b)(6 + 5b) - 3(1 + b)\left((3 + b + A)\frac{R_{01}}{A} - (3 + b + B)\frac{R_{9-10}}{B}\right)\right]. \tag{B7}$$

When we already have R_{01} and R_{9-10} using the methods presented above, then we also have the expression for R_{09} using equation (B7). This equation can also be generalized iteratively when adding more intermediate frames. With only one intermediate frame (section 5), the corresponding relation is obtained from $R_{06} = 8(V_{06} + \tilde{V}_{06})/9$ and equation (25):

$$R_{06} = \frac{1}{b}\left[\frac{2}{3}(6 + 5b) - (3 + 2b + A)\frac{R_{01}}{A} - (3 + 2b + B)\frac{R_{67}}{B}\right]. \tag{B8}$$

As we did above, in the case of one intermediate frame, we can also solve explicitly equation (B2) to obtain the expression of α:

$$\alpha = \frac{4}{11}\frac{9(1 + b)(3 + b) + (15 + 17b + 3b^2)(A + B) + (7 + 6b)AB}{9(1 + b)(3 + b) + (9 + 9b + b^2)(A + B) + (3 + 2b)AB}, \tag{B9}$$

and we obtain:

$$R_{09} = \frac{9(1 + b)(3 + b) + (15 + 17b + 3b^2)(A + B) + (7 + 6b)AB}{9(1 + b)(3 + b) + (9 + 9b + b^2)(A + B) + (3 + 2b)AB}. \tag{B10}$$

In the simplest case, we have $A = B = b$, giving:

$$R_{09}(4) = \frac{9 + 16b + 6b^2}{9 + 12b + 2b^2}. \tag{B11}$$

In the case of an extended cylinder network, equation (B10) can be written:

$$R_{09}(m) = \frac{12(1+b)(3+b) + 2(15+17b+3b^2)(R_{01}(q)+R_{01}(p)) + 3(7+6b)R_{01}(q)R_{01}(p)}{12(1+b)(3+b) + 2(9+9b+b^2)(R_{01}(q)+R_{01}(p)) + 3(3+2b)R_{01}(q)R_{01}(p)}, \quad \text{(B12)}$$

and the limit for large q and p is, with $R_{01}(p) = R_{01}(q) = R_{01}(\infty)$, using equation (8):

$$R_{i(i+9)}(\infty) = \frac{1}{b^2}\left(6 + 8b + 3b^2 - 18\frac{(1+b)^2}{3+4b}\sqrt{1+\frac{4b}{3}}\right). \quad \text{(B13)}$$

As in section 5, we can also derive resistances in $m \times 3$ globe networks. To get the expression of $\widehat{R}_{1-10}(5)$, we apply equation (B12) with $A = B = b'$ and we get:

$$\widehat{R}_{1-10}(5) = \frac{9+19b+9b^2}{9+15b+5b^2}. \quad \text{(B14)}$$

We can also obtain the expression of the internal resistance across two frames in the $m \times 3$ globe network. We replace in equation (B12) $R_{01}(q)$ and $R_{01}(p)$, by $\hat{R}_{01}(m_1 - 1)$ and $\hat{R}_{01}(m_2 - 1)$, respectively, and then, using equation (8), we obtain:

$$\widehat{R}_{i(i+9)}(m) = \frac{12(1+b)(3+b) + 2(6+5b)(\widehat{R}_{01}(m_1)+\widehat{R}_{01}(m_2)) - b(4+3b)\widehat{R}_{01}(m_1)\widehat{R}_{01}(m_2)}{12(1+b)(3+b) - 2b(3+2b)(\widehat{R}_{01}(m_1)+\widehat{R}_{01}(m_2)) + b^2\widehat{R}_{01}(m_1)\widehat{R}_{01}(m_2)}. \quad \text{(B15)}$$

Using equations (B12) and (B15), we can obtain $R_{3-12}(5)$, $R_{3-12}(6)$, $\widehat{R}_{4-13}(6)$, $\widehat{R}_{4-13}(7)$ and so on.

Appendix C

Comparison with the results of Tan's recursion-transform method

Our results for the $m \times 3$ cylinder (table 1) and globe (table 2) networks can be compared with the results of Tan's RT method, explicitly given in this case [29]. Let us first consider the $m \times 3$ cylinder network.

Following the notations used by Tan and Tan in [29], the $m \times 3$ cylinder network is made of $n = m - 1$ triangular frames $A_iB_iC_i$ with $0 \leqslant i \leqslant n$ of resistors r_0 and the $n + 1$ frames are connected by equal resistors r. Here, we consider the two-point resistances between node A_0 and all other nodes, given by the following concise and elegant general expressions:

$$\begin{cases} R_{\Delta\times n}(A_0, A_k) = \dfrac{k}{3} + \dfrac{2}{9h}\dfrac{\Delta F_n - 2\Delta F_{n-k} + \Delta F_k \Delta F_{n-k}}{F_{n+1}} \\ R_{\Delta\times n}(A_0, B_k) = \dfrac{k}{3} + \dfrac{2}{9h}\dfrac{\Delta F_n + \Delta F_{n-k} + \Delta F_k \Delta F_{n-k}}{F_{n+1}} \end{cases}, \quad \text{(C1)}$$

where $h = r/r_0$ [29], corresponding to $h = 1/b$ in our notations. Equation (C1) corresponds to equations (104) and (107), respectively, of page 11 in [29]. To compare with our results, we have expressed the two-points resistances in equation (C1) in units of r and simplified the notations. The recursive coefficients in equation (C1) are given by equation (4) of [29] as:

$$\begin{cases} \Delta F_k = F_{k+1} - F_k \\ F_k = \dfrac{\lambda^k - \bar{\lambda}^k}{\lambda - \bar{\lambda}} \end{cases}, \quad \text{(C2)}$$

where λ and $\bar{\lambda}$are the roots of Tan's characteristic equation. They are given by equation (101) of [29]:

$$\begin{cases} \lambda = 1 + \dfrac{3}{2}h + \sqrt{\left(1+\dfrac{3}{2}h\right)^2 - 1} \\ \bar{\lambda} = 1 + \dfrac{3}{2}h - \sqrt{\left(1+\dfrac{3}{2}h\right)^2 - 1} \end{cases}. \quad \text{(C3)}$$

From equation (C3), we have the property that $\lambda\bar{\lambda} = 1$.

To obtain explicit expressions from equation (C1), we calculate some values of the recursive parameters F_k and ΔF_k. We have $F_0 = 0, F_1 = 1$ and $\Delta F_0 = 1$. For $k = 2$, we have:

$$\begin{cases} F_2 = \dfrac{\lambda^2 - \bar{\lambda}^2}{\lambda - \bar{\lambda}} = \lambda + \bar{\lambda} = 2 + 3h \\ \Delta F_1 = F_2 - F_1 = 1 + 3h \end{cases}. \tag{C4}$$

For $k = 3$, we have:

$$\begin{cases} F_3 = \dfrac{\lambda^3 - \bar{\lambda}^3}{\lambda - \bar{\lambda}} = \lambda^2 + \lambda\bar{\lambda} + \bar{\lambda}^2 = (\lambda + \bar{\lambda})^2 - \lambda\bar{\lambda} = 3(1 + 4h + 3h^2) = 3(1 + h)(1 + 3h) \\ \Delta F_3 = F_3 - F_2 = 1 + 9h + 9h^2 \end{cases}. \tag{C5}$$

Similarly:

$$\begin{cases} F_4 = \dfrac{\lambda^4 - \bar{\lambda}^4}{\lambda - \bar{\lambda}} = \lambda^3 + \lambda^2\bar{\lambda} + \lambda\bar{\lambda}^2 + \bar{\lambda}^3 = (\lambda + \bar{\lambda})[(\lambda + \bar{\lambda})^2 - 2] = (2 + 3h)(2 + 12h + 9h^2) \\ \Delta F_4 = F_4 - F_3 = 1 + 18h + 45h^2 + 27h^3 \end{cases}. \tag{C6}$$

and:

$$\begin{cases} F_5 = \dfrac{\lambda^5 - \bar{\lambda}^5}{\lambda - \bar{\lambda}} = \lambda^4 + \lambda^3\bar{\lambda} + \lambda^2\bar{\lambda}^2 + \lambda\bar{\lambda}^3 + \bar{\lambda}^4 = (\lambda + \bar{\lambda})^4 - 3(\lambda + \bar{\lambda})^2 + 1 \\ = 5 + 60h + 189h^2 + 216h^3 + 81h^4 \\ \Delta F_5 = F_5 - F_4 = 1 + 30h + 135h^2 + 189h^3 + 81h^4 \end{cases}. \tag{C7}$$

Now, applying equation (C1), we get, for $m = 2$ ($n = 1$):

$$\begin{cases} R_{01}(2) = R_{\Delta\times 1}(A_0, B_0) = \dfrac{2}{3h}\dfrac{\Delta F_1}{F_2} = \dfrac{2}{3h}\dfrac{1 + 3h}{2 + 3h} \\ R_{03}(2) = R_{\Delta\times 1}(A_0, A_1) = \dfrac{1}{3} + \dfrac{4}{9h}\dfrac{\Delta F_1 - 1}{F_2} = \dfrac{2 + h}{2 + 3h} \\ R_{04}(2) = R_{\Delta\times 1}(A_0, B_1) = \dfrac{1}{3} + \dfrac{2}{9h}\dfrac{2\Delta F_1 + 1}{F_2} = \dfrac{2 + 6h + 3h^2}{3h(2 + 3h)} \end{cases}. \tag{C8}$$

For $m = 3$ ($n = 2$), we obtain:

$$\begin{cases} R_{01}(3) = R_{\Delta\times 2}(A_0, B_0) = \frac{2}{3h}\frac{\Delta F_2}{F_3} = \frac{2}{9h}\frac{1 + 9h + 9h^2}{(1 + h)(1 + 3h)} \\ R_{03}(3) = R_{\Delta\times 2}(A_0, A_1) = \frac{1}{3} + \frac{2}{9h}\frac{\Delta F_2 - 2\Delta F_1 + \Delta F_1{}^2}{F_3} = \frac{3 + 8h + 3h^2}{3(1 + h)(1 + 3h)} \end{cases}. \tag{C9}$$

and:

$$\begin{cases} R_{04}(3) = R_{\Delta\times 2}(A_0, B_1) = \frac{1}{3} + \frac{2}{9h}\frac{\Delta F_2 + \Delta F_1 + \Delta F_1{}^2}{F_2} = \frac{2 + 15h + 24h^2 + 9h^3}{9h(1 + h)(1 + 3h)} \\ R_{06}(3) = R_{\Delta\times 2}(A_0, A_2) = \frac{2}{3} + \frac{4}{9h}\frac{\Delta F_2 - 1}{F_3} = 2\frac{1 + h}{1 + 3h} \\ R_{07}(3) = R_{\Delta\times 2}(A_0, B_2) = \frac{2}{3} + \frac{2}{9h}\frac{2\Delta F_2 + 1}{F_3} = \frac{2}{9}\frac{1 + 9h + 18h^2 + 9h^3}{h(1 + h)(1 + 3h)} \end{cases}. \tag{C10}$$

Finally, for R_{01}, we obtain, for $m = 4$ and $m = 5$:

$$\begin{cases} R_{01}(4) = R_{\Delta\times 3}(A_0, B_0) = \frac{2}{3h}\frac{\Delta F_3}{F_4} = \frac{2}{3h}\frac{1 + 18h + 45h^2 + 27h^3}{(2 + 3h)(2 + 12h + 9h^2)} \\ R_{01}(5) = R_{\Delta\times 4}(A_0, B_0) = \frac{2}{3h}\frac{\Delta F_4}{F_5} = \frac{2}{3h}\frac{1 + 30h + 135h^2 + 189h^3 + 81h^4}{5 + 60h + 189h^2 + 216h^3 + 81h^4} \end{cases}. \tag{C11}$$

The expressions from equations (C8)–(C11) coincide, when replacing h by $1/b$, with the results given in table 1 and equations (30) and (34) of section 5.

Examples of expressions can also be obtained for the $m \times 3$ globe network. Using equation (103) of [29], we have:

$$\widehat{R}_{01}(n) = R_{\Delta\times n}(A_0, A_1) = \frac{1}{3} + \frac{2}{9h}\frac{\alpha_1\alpha_{n-1}}{G_n}, \tag{C12}$$

where G_n and α_n are given by equations (7) and (5) of [29], respectively:

$$\begin{cases} G_n = F_{n+1} - 2F_n + F_{n-1} \\ \alpha_n = \Delta F_n - \Delta F_{n-1} \end{cases}. \tag{C13}$$

Using the results (C4-C7) given above, we have:

$$\begin{cases} G_2 = F_3 - 2F_2 + F_1 = 3h(2+3h) \\ G_3 = F_4 - 2F_3 + F_2 = 9h(1+h)(1+3h) \\ G_4 = F_5 - 2F_4 + F_3 = 3h(2+3h)(2+12h+9h^2) \end{cases}, \tag{C14}$$

and:

$$\begin{cases} \alpha_1 = \Delta F_1 - \Delta F_0 = 3h \\ \alpha_2 = \Delta F_2 - \Delta F_1 = 3h(2+3h) \\ \alpha_3 = \Delta F_3 - \Delta F_2 = 9h(1+h)(1+3h) \end{cases}. \tag{C15}$$

Thus, we obtain the following explicit results for two-point resistances in the $m \times 3$ globe network:

$$\begin{cases} \widehat{R}_{01}(2) = \dfrac{1}{3} + \dfrac{2}{9h}\dfrac{\alpha_1^2}{G_2} = \dfrac{4+3h}{3(2+3h)} \\ \widehat{R}_{01}(3) = \dfrac{1}{3} + \dfrac{2}{9h}\dfrac{\alpha_1\alpha_2}{G_3} = \dfrac{7+18h+9h^2}{9(1+h)(1+3h)} \\ \widehat{R}_{01}(4) = \dfrac{1}{3} + \dfrac{2}{9h}\dfrac{\alpha_1\alpha_3}{G_4} = \dfrac{10+54h+72h^2+27h^3}{3(2+3h)(2+12h+9h^2)} \end{cases}. \tag{C16}$$

The expressions from equation (C16) coincide, when replacing h by $1/b$, with the results given in table 2. Incidentally, using equation (1), we have:

$$\widehat{R}_{12}(n) = 3\widehat{R}_{01}(n) - 1 = \frac{2}{3h}\frac{\alpha_1\alpha_{n-1}}{G_n}. \tag{C17}$$

ORCID iDs

Frédéric Perrier https://orcid.org/0000-0002-6306-7478
Frédéric Girault https://orcid.org/0000-0001-8470-1792

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