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Donato Bini, Guillaume Faye, Andrea Geralico

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Donato Bini
Istituto per le Applicazioni del Calcolo "M. Picone," CNR, I-00185 Rome, Italy

Guillaume Faye
G$\mathbb{R}\varepsilon\mathbb{C}$O, Institut d'Astrophysique de Paris, UMR 7095 CNRS, Sorbonne Universités, UPMC Univ Paris 06, F-75014 Paris, France

Andrea Geralico
Istituto per le Applicazioni del Calcolo "M. Picone," CNR, I-00185 Rome, Italy
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The features of equatorial motion of an extended body in Kerr spacetime are investigated in the framework of the Mathisson-Papapetrou-Dixon model. The body is assumed to stay at quasiequilibrium and respond instantly to external perturbations. Besides the mass, it is completely determined by its spin, the multipolar expansion being truncated at the quadrupole order, with a spin-induced quadrupole tensor. The study of the radial effective potential allows us to analytically determine the innermost stable circular orbit shift due to spin and the associated frequency of the last circular orbit.

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I. INTRODUCTION

The description of the gravitational interaction between the constituents of a binary system in the general theory of relativity requires taking into due account their internal structure. The orbital dynamics of two bound compact objects is tackled in the literature by plenty of different methods resorting to various approximation schemes. Analytic approaches include notably the post-Newtonian approximation [1], possibly implemented using effective field theory techniques [2], as well as the gravitational self-force corrections to geodesic motion [3], which can both be combined efficiently by means of the "effective-one-body" approach [4,5].

When the mass of one body is much smaller than the other, the problem boils down to studying the dynamics of an extended body in a fixed background field, generated by the heavier mass. In this approximation, a self-consistent model describing the evolution of both linear and angular momenta for pole-dipole sources was developed by Mathisson [6], Papapetrou [7,8], Pirani [9], Tulczyjew [10], and later generalized to bodies endowed with higher multipoles by Dixon [11–15]. The Mathisson-Papapetrou-Dixon (MPD) model accounts for the motion, on a fixed background, of a point-size test object with internal degrees of freedom, in the absence of significant gravitational backreaction. Bailey and Israel [16] have shown that the corresponding equations are actually derivable from an effective Lagrangian. A Hamiltonian scheme has also been designed in Ref. [17] in the case of purely spinning bodies deprived of any higher-multipole structure.

The main astrophysical situation for which a fully relativistic treatment is needed occurs when the object—assumed to be compact to avoid tidal disruption—experiences the strong field produced by a nearby black hole. The MPD approach may then be used to investigate the evolution of the system, but the parameters of the model must be regarded as effective ones [18]. Although self-force effects are not negligible on larger-than-orbital time scales, they only yield higher-order corrections. Furthermore, combined with dimensional regularization, the MPD model is appropriate to describe the dynamics of (self-gravitating) compact binaries including spins and associated finite-size effects, in the post-Newtonian framework [19–21]. It yields results that are dynamically equivalent, up to the third post-Newtonian order at least, to those derived from suitable effective actions [22–26] computed with methods building on the early attempt [27] (see Ref. [28] for an extensive presentation of a fully consistent Lagrangian reduction valid to all orders in spins). The post-Newtonian reduction of appropriate Arnowitt-Deser-Misner Hamiltonians leads again to the same dynamics [29–31], as reported with more detail in [32]. On the other hand, the MPD evolution is entirely encoded into a "skeleton" stress-energy tensor, known explicitly at the quadrupolar level [33]. The octupolar contributions have also been obtained recently [34] assuming an effective, Bailey and Israel–type, Lagrangian [16]. Those corresponding explicit expressions have been used to build accurate theoretical templates for the signal of gravitational radiation emitted by those sources [21,34–36], in the context of the data analysis of gravitational-wave observatories, such as the advanced Virgo [37] and LIGO [38] detectors, the future cryogenic interferometer KAGRA

[39] or, possibly, the space-based observatory eLISA [40], a candidate for the future L3 mission of the European Space Agency. Most of the post-Newtonian expressions can be checked by comparing them to the test-body counterparts, in the extreme mass-ratio limit.

In this paper we study the dynamics of an extended body endowed with both spin and quadrupole moment in a Kerr spacetime using the MPD model. The motion is assumed to be confined on the equatorial plane, the spin vector of the body being aligned with the axis of rotation of the central object. In previous works, we have discussed the effects on the dynamics of a general quadrupole tensor in both Schwarzschild and Kerr spacetimes [41–43]. Here we consider more specifically the case of a spin-induced quadrupole tensor. We assume that the object reaches thermodynamic equilibrium in its proper frame on time scales that are very short compared with the orbital period and neglect the tidal deformations. Its internal state, thus, depends adiabatically on the mass and the (instantaneous) spin. Using effective field theory arguments, it is then straightforward to check that the body quadrupole is actually quadratic in the spin. This situation was described in detail by Steinhoff and Puetzfeld [44], who developed a very general framework to include quadratic in spin corrections as well as tidal interactions in the MPD scheme, with special attention to the study of the binding energy of the system as obtained from the analysis of the associated effective potential. Later, Hinderer *et al.* [45] performed an analysis of the corresponding dynamics, in order to compare the periastron advance and precession frequencies with those of a different approach, but restricted themselves to a very special (although physically motivated) choice of the quadrupole tensor, leading to great simplifications in the analytic treatment.

Here we shall assume the same form of the quadrupole tensor as in Ref. [44], but neglect quadrupolar tides, i.e., our quadrupole tensor is of the electric type only and is proportional to the trace-free part of the square of the spin tensor by a constant parameter, whose numerical value is a property of the body under consideration. For neutron stars, such a quantity depends on the equation of state [46], while it is exactly 1 for black holes. We keep it a free parameter of the model that can affect associated observables, like the energy and the angular momentum, which we computed explicitly and compared with the results of Ref. [45], or the innermost stable circular orbit (ISCO) and its frequency, discussed here in detail. We achieve a fully analytic treatment of the MPD equations in a "perturbative" scheme, obtaining corrections to geodesic motion up to the second order in spin.

Throughout this work we use geometrical units, setting the Newton constant G and the speed of light c to 1. Tensors are represented either in abstract notation or in index notation combined with the Einstein convention, depending on the context. Greek indices refer to spacetime coordinates and vary from 0 to 3, i.e., $\mu = 0, 1, 2, 3$, whereas Latin indices, ranging from 1 to 3, label space coordinates. The notation ∂_μ stands indifferently for the partial derivative with respect to the μth coordinate x^μ or for the coordinate basis vector associated to x^μ, while d denotes the exterior derivative. The spacetime metric $g_{\mu\nu}$, taken to have signature $(-,+,+,+)$, defines a unique Levi-Civita covariant derivative ∇_μ and an associated Riemann curvature $R^\mu{}_{\nu\alpha\beta}$, with the convention that $R^\mu{}_{\nu\alpha\beta}v^\nu = (\nabla_\alpha\nabla_\beta - \nabla_\beta\nabla_\alpha)v^\mu$ for any vector field v^μ. Symmetrization of a tensor T over a set indices is indicated by round brackets enclosing them: $T^{(\mu\nu)} = (T^{\mu\nu} + T^{\nu\mu})/2$. Instead, for index antisymmetrization, square brackets are used: $T^{[\mu\nu]} = (T^{\mu\nu} - T^{\nu\mu})/2$.

II. MPD DESCRIPTION OF EXTENDED BODIES

In the quadrupole approximation, the MPD equations read

$$\frac{\mathrm{D}P^\mu}{\mathrm{d}\tau} = -\frac{1}{2}R^\mu{}_{\nu\alpha\beta}U^\nu S^{\alpha\beta} - \frac{1}{6}J^{\alpha\beta\gamma\delta}\nabla^\mu R_{\alpha\beta\gamma\delta} \equiv F^\mu_{(\mathrm{spin})} + F^\mu_{(\mathrm{quad})}, \tag{2.1}$$

$$\frac{\mathrm{D}S^{\mu\nu}}{\mathrm{d}\tau} = 2P^{[\mu}U^{\nu]} + \frac{4}{3}J^{\alpha\beta\gamma[\mu}R^{\nu]}{}_{\gamma\alpha\beta} \equiv D^{\mu\nu}_{(\mathrm{spin})} + D^{\mu\nu}_{(\mathrm{quad})}, \tag{2.2}$$

where P^μ is the total 4-momentum of the body with mass $m \equiv (-P^\mu P_\mu)^{1/2}$ and direction $u^\mu \equiv P^\mu/m$ (hence, $P^\mu = mu^\mu$ and $u \cdot u \equiv u^\mu u_\mu = -1$ by definition), $S^{\mu\nu}$ is the (antisymmetric) spin tensor, $J^{\alpha\beta\gamma\delta}$ is the quadrupole tensor, and $U^\mu = \mathrm{d}z^\mu/\mathrm{d}\tau$ is the timelike unit tangent vector—or 4-velocity—of the body "reference" line, parametrized by the proper time τ [with parametric equations $x^\mu = z^\mu(\tau)$], used to make the multipole reduction.

In order to ensure that the model is mathematically self-consistent, the reference point in the object should be specified by imposing some additional conditions. Here we shall take the Tulczyjew conditions [10,11],

$$S^{\mu\nu}u_\nu = 0. \tag{2.3}$$

With this choice, the spin tensor can be fully represented by a spatial vector (with respect to u),

$$S(u)^\alpha = \frac{1}{2}\eta(u)^\alpha{}_{\beta\gamma}S^{\beta\gamma} = [^{*(u)}S]^\alpha, \tag{2.4}$$

where $\eta(u)_{\alpha\beta\gamma} = \eta_{\mu\alpha\beta\gamma}u^\mu$ is the spatial unit volume 3-form (with respect to u) built from the 4-volume form $\eta_{\alpha\beta\gamma\delta} = \sqrt{-g}\epsilon_{\alpha\beta\gamma\delta}$, with $\epsilon_{\alpha\beta\gamma\delta}$ $(\epsilon_{0123} = 1)$ being the Levi-Civita alternating symbol and g the determinant of the metric in a generic coordinate grid. Using a fairly standard convention, hereafter we denote the spacetime dual of a tensor S (such that ${}^*S^{\alpha\beta} = \eta^{\alpha\beta}{}_{\gamma\delta}S^{\gamma\delta}/2$) by ${}^*S$, whereas the spatial dual of a spatial tensor S with respect to u is represented by ${}^{*(u)}S$. It is also useful to introduce the signed magnitude s of the spin vector, which is not constant in general along the trajectory of the extended body,

$$s^2 = S(u)^\beta S(u)_\beta = \frac{1}{2}S_{\mu\nu}S^{\mu\nu} = -\frac{1}{2}\mathrm{Tr}[S^2], \tag{2.5}$$

with $\mathrm{Tr}[S^2] = -S_{\mu\nu}S^{\mu\nu}$.

A. Spin-induced quadrupole tensor

The $1+3$ decomposition of the quadrupole tensor J and its general properties are briefly reviewed in Appendix A. We will consider here the physically relevant case where it is completely determined by the instantaneous spin structure of the body (see, e.g., Refs. [44,47]). More specifically, we shall let the quadrupole tensor have the form

$$J^{\alpha\beta\gamma\delta} = 4u^{[\alpha}\widetilde{\mathcal{X}}(u)^{\beta][\gamma}u^{\delta]}, \tag{2.6}$$

with

$$\widetilde{\mathcal{X}}(u) = \frac{3}{4}\frac{C_Q}{m}[S^2]^{\mathrm{STF}}, \tag{2.7}$$

where C_Q is a "polarizability" constant and $[S^2]^{\mathrm{STF}}$ denotes the trace-free part of $S^{\alpha\mu}S_\mu{}^\beta$, i.e., in terms of both the spin vector and the associated spin invariant,

$$\begin{aligned}[S^2]^{\mathrm{STF}\alpha\beta} &= S^{\alpha\mu}S_\mu{}^\beta - \frac{1}{3}P(u)^{\alpha\beta}S_{\rho\sigma}S^{\sigma\rho} \\ &= S(u)^\alpha S(u)^\beta - \frac{1}{3}s^2P(u)^{\alpha\beta}. \end{aligned}\tag{2.8}$$

The values of C_Q associated with compact objects are given, e.g., in Ref. [32]. The normalization is such that $C_Q = 1$ in the case of a black hole [48], whereas for neutron stars C_Q depends on the equation of state and varies roughly between 4 and 8 [46].

For the spin-induced quadrupole tensor (2.6), the link between P_μ and U_μ takes a particularly simple form if cubic-in-spin corrections are neglected. Indeed, contraction of Eq. (2.2) with P_ν shows that, apart from corrections of order $J = \mathcal{O}(S^2)$, the difference $(P\cdot U)P^\mu - P^2U^\mu$ is precisely $P_\nu \mathrm{D}S^{\mu\nu}/\mathrm{d}\tau = -S^{\mu\nu}\mathrm{D}P_\nu/\mathrm{d}\tau$, which can be seen to be of order $\mathcal{O}(S^2)$ from Eq. (2.1). In the end, P^μ is approximately proportional to U^μ and, as an important consequence, the right-hand side of the precession equation (2.2) is at least quadratic in the spin. As, on the other hand, the right-hand side of the precession equation (2.2) is at least linear in the spin, we conclude that the time differentiation of our kinematical variables actually multiply any combination of them by a factor $\mathcal{O}(S)$.

In Ref. [45], the MPD description of test bodies endowed with a spin-induced quadrupolar structure is used to check the consistency of the computation of the periastron advance for a binary system with the effective-one-body formalism in the extreme mass-ratio limit, including terms that are quadratic in the spin. The quadrupole tensor is assumed there to be that of a black hole and to take the form (2.6) with $C_Q = 1$, so that we must recover the results of Ref. [45] for this particular value.

B. Simplified form of the MPD equations

The MPD equations can be written in a more convenient form at quadratic order in the spin. First, the quadrupolar contribution $F^\mu_{(\mathrm{quad})}$ in Eq. (2.1) splits into a parallel and a perpendicular part with respect to the direction U^μ:

$$F^\mu_{(\mathrm{quad})} = -\frac{1}{6}J^{\alpha\beta\gamma\delta}[P(U)]^{\mu\nu}\nabla_\nu R_{\alpha\beta\gamma\delta} + \frac{1}{6}J^{\alpha\beta\gamma\delta}U^\mu\frac{\mathrm{D}}{\mathrm{d}\tau}R_{\alpha\beta\gamma\delta}. \tag{2.9}$$

Next, neglecting remainders that are cubic in the spin, the quadrupole $J^{\alpha\beta\gamma\delta}$ and the velocity U^μ in the last term may be moved under the operator $\mathrm{D}/\mathrm{d}\tau$, since their covariant time differentiation would actually produce terms smaller than the original ones by a factor $\mathcal{O}(S)$, as explained in the previous subsection. The equations of motion then become

$$\begin{aligned}\frac{\mathrm{D}P^\mu}{\mathrm{d}\tau} = &-\frac{1}{2}R^\mu{}_{\nu\alpha\beta}U^\nu S^{\alpha\beta} - \frac{1}{6}J^{\alpha\beta\gamma\delta}[P(U)]^{\mu\nu}\nabla_\nu R_{\alpha\beta\gamma\delta} \\ &+ \frac{\mathrm{D}}{\mathrm{d}\tau}(m_J U^\mu), \end{aligned}\tag{2.10}$$

where we have posed

$$m_J = \frac{1}{6}J^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}. \tag{2.11}$$

Thus, defining a modified linear momentum $p^\mu = P^\mu - m_J U^\mu$ effectively changes the quadrupolar force $F^\mu_{(\mathrm{quad})}$ into its projection $[P(U)]^\mu{}_\nu F^\nu_{(\mathrm{quad})}$ orthogonal to U^μ, while the spin force $F^\mu_{(\mathrm{spin})}$ is orthogonal to U^μ. The precession equations are unaffected. More explicitly, Eqs. (2.1) and (2.2) expressed in terms of p^μ become

$$\frac{\mathrm{D}p^\mu}{\mathrm{d}\tau} = -\frac{1}{2}R^\mu{}_{\nu\alpha\beta}U^\nu S^{\alpha\beta} - \frac{1}{6}J^{\alpha\beta\gamma\delta}\nabla^\perp R_{\alpha\beta\gamma\delta} + \mathcal{O}(S^3), \tag{2.12a}$$

$$\frac{\mathrm{D}S^{\mu\nu}}{\mathrm{d}\tau} = 2p^{[\mu}U^{\nu]} + \frac{4}{3}J^{\alpha\beta\gamma[\mu}R^{\nu]}{}_{\gamma\alpha\beta} + \mathcal{O}(S^3),$$

with $\nabla^\perp \equiv [P(U)]^{\mu\nu}\nabla_\nu$. Now, if we multiply the equations of motion (2.12a) by $p_\mu \propto U_\mu + \mathcal{O}(S^2)$, we see that, at our accuracy level, the first term on the right-hand side vanishes by virtue of the Riemann-tensor symmetries, whereas the second one is zero due to the contraction of U_μ with

$[P(U)]^{\mu\nu}$. We conclude that the effective mass $(-p_\mu p^\mu)^{1/2}$ is conserved, modulo cubic spin corrections (see also Appendix B). It may be regarded as the "bare" mass of the extended body, m_0, at the quadratic order in the spin, so that its mass m to order $\mathcal{O}(S^2)$ is given by

$$m = m_0 + m_J + \mathcal{O}(S^3). \tag{2.13}$$

Finally, the MPD equations (2.1)–(2.3) imply that the unit vectors U and u are related by

$$u^\mu = U^\mu + \frac{1}{m_0} D^{\mu\nu}_{(\mathrm{quad})} U_\nu + \frac{1}{m_0^2} S^{\mu\nu} F_{(\mathrm{spin})_\nu} + \mathcal{O}(S^3). \tag{2.14}$$

C. Conserved quantities

In stationary and axisymmetric spacetimes endowed with Killing symmetries, the energy E and the total angular momentum J are conserved quantities along the motion, associated with the timelike Killing vector $\xi = \partial_t$ and the azimuthal Killing vector $\eta = \partial_\phi$, respectively. They are given by

$$\begin{aligned} E &= -\xi_\alpha P^\alpha + \frac{1}{2} S^{\alpha\beta} \nabla_\beta \xi_\alpha, \\ J &= \eta_\alpha P^\alpha - \frac{1}{2} S^{\alpha\beta} \nabla_\beta \eta_\alpha, \end{aligned} \tag{2.15}$$

where $\nabla_\beta \xi_\alpha = g_{t[\alpha,\beta]}$ and $\nabla_\beta \eta_\alpha = g_{\phi[\alpha,\beta]}$. Note that E and J as defined above are conserved to all multipole orders in spite of the higher multipolar structure of the body, which is entirely encoded in P [49].

The conserved quantities (2.15) for a purely dipolar particle in a Kerr spacetime have been computed, e.g., in Refs. [50,51]. The expressions given there are general enough to account for all higher-order spin corrections but those coming from the spin-induced multipole moments (i.e., are exact when the quadrupole and higher multipole moments vanish). This means in practice that our results should reduce to those of Refs. [50,51] for $C_Q = 0$.

III. MOTION IN A KERR SPACETIME

The Kerr metric in standard Boyer-Lindquist coordinates (t, r, θ, ϕ) reads

$$\mathrm{d}s^2 = -\left(1 - \frac{2Mr}{\Sigma}\right)\mathrm{d}t^2 - \frac{4aMr}{\Sigma}\sin^2\theta \mathrm{d}t\mathrm{d}\phi + \frac{\Sigma}{\Delta}\mathrm{d}r^2 + \Sigma \mathrm{d}\theta^2 + \frac{(r^2+a^2)^2 - \Delta a^2 \sin^2\theta}{\Sigma}\sin^2\theta \mathrm{d}\phi^2, \tag{3.1}$$

with $\Delta = r^2 - 2Mr + a^2$ and $\Sigma = r^2 + a^2\cos^2\theta$. Here, $a \geq 0$ and $M \geq a$ denote the specific angular momentum and the total mass of the spacetime solution, respectively. The event and inner horizons are located at $r_\pm = M \pm \sqrt{M^2 - a^2}$.

Let us introduce the zero angular-momentum observer (ZAMO) family of fiducial observers, with 4-velocity

$$n = N^{-1}(\partial_t - N^\phi \partial_\phi) \tag{3.2}$$

orthogonal to the hypersurfaces of constant t, where $N = (-g^{tt})^{-1/2}$ and $N^\phi = g_{t\phi}/g_{\phi\phi}$ are the lapse and shift functions, respectively. A suitable spatial orthonormal frame adapted to ZAMOs is given by

$$\begin{aligned} e_{\hat{t}} &= n, & e_{\hat{r}} &= \frac{1}{\sqrt{g_{rr}}}\partial_r \equiv \partial_{\hat{r}}, \\ e_{\hat{\theta}} &= \frac{1}{\sqrt{g_{\theta\theta}}}\partial_\theta \equiv \partial_{\hat{\theta}}, & e_{\hat{\phi}} &= \frac{1}{\sqrt{g_{\phi\phi}}}\partial_\phi \equiv \partial_{\hat{\phi}}, \end{aligned} \tag{3.3}$$

with dual

$$\begin{aligned} \omega^{\hat{t}} &= N\mathrm{d}t, & \omega^{\hat{r}} &= \sqrt{g_{rr}}\mathrm{d}r, \\ \omega^{\hat{\theta}} &= \sqrt{g_{\theta\theta}}\mathrm{d}\theta, & \omega^{\hat{\phi}} &= \sqrt{g_{\phi\phi}}(\mathrm{d}\phi + N^\phi \mathrm{d}t). \end{aligned} \tag{3.4}$$

The ZAMOs are subject to the acceleration $a(n) = \nabla_n n$. They are locally nonrotating, in the sense that their vorticity vector $\omega(n)^\alpha$ vanishes due to their surface-orthogonal character, but they have a nonzero trace-free expansion tensor $\theta(n)_{\alpha\beta} \equiv P(n)^\mu{}_\alpha P(n)^\nu{}_\beta \nabla_{(\mu} n_{\nu)}$; the latter, in turn, can be completely described by an expansion vector $\theta_{\hat{\phi}}(n)^\alpha = \theta(n)^\alpha{}_\beta e_{\hat{\phi}}{}^\beta$, such that

$$\theta(n) = e_{\hat{\phi}} \otimes \theta_{\hat{\phi}}(n) + \theta_{\hat{\phi}}(n) \otimes e_{\hat{\phi}}, \tag{3.5}$$

where $\otimes$ represents the tensor product. The nonzero ZAMO kinematical quantities (i.e., acceleration and expansion) all belong to the r-θ 2-plane of the tangent space [52–55], with

$$\begin{aligned} a(n) &= a(n)^{\hat{r}} e_{\hat{r}} + a(n)^{\hat{\theta}} e_{\hat{\theta}} \equiv \partial_{\hat{r}}(\ln N) e_{\hat{r}} + \partial_{\hat{\theta}}(\ln N) e_{\hat{\theta}}, \\ \theta_{\hat{\phi}}(n) &= \theta_{\hat{\phi}}(n)^{\hat{r}} e_{\hat{r}} + \theta_{\hat{\phi}}(n)^{\hat{\theta}} e_{\hat{\theta}} \\ &\equiv -\frac{\sqrt{g_{\phi\phi}}}{2N}(\partial_{\hat{r}} N^\phi e_{\hat{r}} + \partial_{\hat{\theta}} N^\phi e_{\hat{\theta}}). \end{aligned} \tag{3.6}$$

It is also useful to introduce the curvature vectors associated with the diagonal metric coefficients,

$$\kappa(x^i,n) = \kappa(x^i,n)^{\hat r} e_{\hat r} + \kappa(x^i,n)^{\hat\theta} e_{\hat\theta} \equiv -[\partial_{\hat r}(\ln\sqrt{g_{ii}})e_{\hat r} + \partial_{\hat\theta}(\ln\sqrt{g_{ii}})e_{\hat\theta}]. \qquad (3.7)$$

We shall use the notation $\kappa(\phi,n)^{\hat r} \equiv k_{(\mathrm{Lie})}$ for the Lie relative curvature [53,54], largely adopted in the literature, and limit our analysis to the equatorial plane ($\theta = \pi/2$) of the Kerr solution, where

$$N = \left[\frac{r\Delta}{r^3 + a^2 r + 2a^2 M}\right]^{1/2}, \qquad N^\phi = -\frac{2aM}{r^3 + a^2 r + 2a^2 M}, \qquad (3.8)$$

and $\Delta = N^2 g_{\phi\phi}$. The ZAMO kinematical quantities as well as the nonvanishing frame components of the Riemann tensor are listed in Appendix C.

Let us now consider a test body rotating in the equatorial plane around the central source. Its 4-velocity U may be written in terms of the velocity $\nu(U,n) = \nu^{\hat r} e_{\hat r} + \nu^{\hat\phi} e_{\hat\phi}$ relative to the ZAMOs, with associated Lorentz factor $\gamma(U,n)$, as

$$U = \gamma(U,n)[n + \nu(U,n)], \qquad \gamma(U,n) = (1 - \|\nu(U,n)\|^2)^{-1/2}. \qquad (3.9)$$

The parametric equations of the orbit are solutions of the evolution equations $U = \mathrm{d}x^\alpha/\mathrm{d}\tau$, i.e.,

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{\gamma}{N}, \qquad \frac{\mathrm{d}r}{\mathrm{d}\tau} = \frac{\gamma\nu^{\hat r}}{\sqrt{g_{rr}}}, \qquad \frac{\mathrm{d}\phi}{\mathrm{d}\tau} = \frac{\gamma}{\sqrt{g_{\phi\phi}}}\left(\nu^{\hat\phi} - \frac{\sqrt{g_{\phi\phi}}N^\phi}{N}\right), \qquad (3.10)$$

where the abbreviated notation $\gamma(U,n) \equiv \gamma$ and $\nu^{\hat a} \equiv \nu(U,n)^{\hat a}$ has been used. For equatorial orbits, a convenient parametrization can be r itself instead of the proper time τ.

A case of particular importance is that of uniform, circular equatorial motion. The unit tangent vector, U, may then be parametrized either by the (constant) angular velocity with respect to infinity ζ or equivalently by the (constant) linear velocity ν with respect to the ZAMOs, i.e.,

$$U = \Gamma[\partial_t + \zeta\partial_\phi] = \gamma[e_{\hat t} + \nu e_{\hat\phi}], \qquad \gamma = (1-\nu^2)^{-1/2}, \qquad (3.11)$$

with

$$\Gamma = [N^2 - g_{\phi\phi}(\zeta + N^\phi)^2]^{-1/2} = \frac{\gamma}{N}, \qquad \zeta = -N^\phi + \frac{N}{\sqrt{g_{\phi\phi}}}\nu. \qquad (3.12)$$

The parametric equations of the orbit reduce to

$$t = t_0 + \Gamma\tau, \qquad r = r_0, \qquad \theta = \frac{\pi}{2}, \qquad \phi = \phi_0 + \Omega\tau, \qquad (3.13)$$

where $\Omega = \Gamma\zeta$ is the proper time orbital angular velocity.

For timelike circular geodesics on the equatorial plane, the expressions of the angular and linear velocities do depend on whether the orbits are corotating (+) or counter-rotating (−). They read

$$\zeta_\pm = \pm\frac{\zeta_K}{1 \pm a\zeta_K}, \qquad \nu_\pm = \frac{r^2\zeta_\pm}{\sqrt{\Delta}}\left(1 + \frac{a^2}{r^2} \mp 2a\zeta_K\right), \qquad (3.14)$$

respectively, with $\zeta_K = \sqrt{M/r^3}$ denoting the Keplerian angular velocity for a nonspinning, Schwarzschild black hole. The Lorentz factor of the corresponding 4-velocity $U_\pm$ is found to be

$$\Gamma_\pm = \frac{\zeta_K}{|\zeta_\pm|}\left(1 - \frac{3M}{r} \pm 2a\zeta_K\right)^{-1/2}. \qquad (3.15)$$

In the static case, we can actually use the Schwarzschild values $\zeta_\pm \to \pm\zeta_K$ and $\nu_\pm \to \pm\nu_K$, with $\nu_K = \sqrt{M/(r-2M)}$. It is convenient [see Eq. (4.21)] to introduce a spacelike unit vector $\bar U_\pm$ that is orthogonal to $U_\pm$ within the Killing 2-plane, by defining

$$\bar U_\pm = \bar\Gamma_\pm[\partial_t + \bar\zeta_\pm\partial_\phi] = \pm\gamma_\pm[\nu_\pm e_{\hat t} + e_{\hat\phi}], \qquad (3.16)$$

in terms of the parameters

$$\bar\Gamma_\pm = \Gamma_\pm|\nu_\pm| = |\Omega_\pm|\frac{r^2}{\sqrt{\Delta}}\left(1 + \frac{a^2}{r^2} \mp 2a\zeta_K\right), \qquad \bar\zeta_\pm = -N^\phi + \frac{N}{\sqrt{g_{\phi\phi}}}\frac{1}{\nu_\pm} = \pm\frac{r\zeta_K}{M}\frac{1 - 2M/r \pm a\zeta_K}{1 + a^2/r^2 \mp 2a\zeta_K}, \qquad (3.17)$$

where $\Omega_\pm = \Gamma_\pm\zeta_\pm$ and where the $\pm$ signs correlate with those of $U_\pm$. Note that $\bar\zeta_\pm = \tilde E_\pm/\tilde L_\pm$ is the ratio between the energy $\tilde E_\pm$ and the azimuthal angular momentum $\tilde L_\pm$ per unit mass of the particle. Those two quantities are expressed as

$$\widetilde{E}_\pm = N\gamma_\pm\left(1+\frac{2aM}{r\sqrt{\Delta}}\nu_\pm\right) = \frac{|\Omega_\pm|}{\zeta_K}\left(1-\frac{2M}{r}\pm a\zeta_K\right),$$
$$\widetilde{L}_\pm = \gamma_\pm\nu_\pm\sqrt{g_{\phi\phi}} = \Omega_\pm r^2\left(1+\frac{a^2}{r^2}\mp 2a\zeta_K\right). \quad (3.18)$$

A. Orbit of the extended body

In order to describe the motion of the extended body according to the MPD model, we need both the timelike unit vector U tangent to the center world line and the unit timelike vector u aligned with the 4-momentum. In the following, we shall assume that the world line of the extended body is confined onto the equatorial plane, so that the 4-velocity U is given by Eq. (3.9), with

$$\nu(U,n)\equiv\nu\hat{\nu}\equiv\nu^{\hat{r}}e_{\hat{r}}+\nu^{\hat{\phi}}e_{\hat{\phi}}=\nu(\cos\alpha e_{\hat{r}}+\sin\alpha e_{\hat{\phi}}), \quad (3.19)$$

where ν and $\alpha\in[0,\frac{\pi}{2}]$ are the signed magnitude of the spatial velocity and its polar angle, measured clockwise from the positive ϕ direction in the r-ϕ tangent plane, respectively, while $\hat{\nu}\equiv\hat{\nu}(U,n)$ is the associated unit vector; hence, ν has positive or negative values for co- or counter-rotating azimuthal motion ($\alpha=\pi/2$) and outward or inward radial motion ($\alpha=0$), with respect to the ZAMOs, respectively.

A similar decomposition holds for the (body) 4-momentum $P=mu$, in the case of equatorial orbits,

$$u=\gamma_u[n+\nu_u\hat{\nu}_u], \qquad \gamma_u=(1-\nu_u^2)^{-1/2}, \quad (3.20)$$

with

$$\hat{\nu}(u,n)\equiv\hat{\nu}_u=\cos\alpha_u e_{\hat{r}}+\sin\alpha_u e_{\hat{\phi}}, \quad (3.21)$$

and $\alpha_u\in[0,\frac{\pi}{2}]$. An orthonormal frame adapted to $u\equiv e_0$ can then be built by introducing the spatial triad:

$$e_1\equiv\hat{\nu}_u^\perp\equiv\sin\alpha_u e_{\hat{r}}-\cos\alpha_u e_{\hat{\phi}},$$
$$e_2=\mathrm{sgn}(\nu_u)\gamma_u(\nu_u n+\hat{\nu}_u), \qquad e_3=-e_{\hat{\theta}}. \quad (3.22)$$

The dual frame of $\{e_\alpha\}$ will be referred to as $\{\omega^\alpha\}$, with $\omega^0=-u^\flat$, $u^\flat$ being the covariant dual of u. The projection of the spin vector into the local rest space of u defines the spin vector $S(u)$ (hereafter simply denoted by S, for short). In the frame (3.22), the spin S decomposes as

$$S=S^1e_1+S^2e_2+S^3e_3. \quad (3.23)$$

B. Setting the body's spin and quadrupole in the aligned case

In the following, we shall consider the special case where the spin vector is aligned with the spacetime rotation axis, i.e.,

$$S=se_3. \quad (3.24)$$

This entails that the spin and quadrupole terms entering the right-hand sides of Eqs. (2.1) and (2.2) decompose, with respect to the frame adapted to u, as

$$F_{(\mathrm{spin})}=F^0_{(\mathrm{spin})}u+F^1_{(\mathrm{spin})}e_1+F^2_{(\mathrm{spin})}e_2,$$
$$F_{(\mathrm{quad})}=F^0_{(\mathrm{quad})}u+F^1_{(\mathrm{quad})}e_1+F^2_{(\mathrm{quad})}e_2, \quad (3.25)$$

and

$$D_{(\mathrm{spin})}=-\omega^0\wedge\mathcal{E}_{(\mathrm{spin})}(u),$$
$$D_{(\mathrm{quad})}=-\omega^0\wedge\mathcal{E}_{(\mathrm{quad})}(u), \quad (3.26)$$

with

$$\mathcal{E}_{(\mathrm{spin})}(u)=\mathcal{E}_{(\mathrm{spin})\,1}\omega^1+\mathcal{E}_{(\mathrm{spin})\,2}\omega^2,$$
$$\mathcal{E}_{(\mathrm{quad})}(u)=\mathcal{E}_{(\mathrm{quad})\,1}\omega^1+\mathcal{E}_{(\mathrm{quad})\,2}\omega^2, \quad (3.27)$$

respectively. The explicit expressions for the above components are listed in Appendix D.

IV. SOLVING THE MPD EQUATIONS FOR NONPRECESSING EQUATORIAL ORBITS

A. Complete set of evolution equations

Under the assumptions of equatorial motion and aligned spins discussed in the previous section, the whole set of MPD equations (2.1)–(2.3) reduces to

$$\frac{\mathrm{d}m}{\mathrm{d}\tau}=F^0_{(\mathrm{spin})}+F^0_{(\mathrm{quad})},$$
$$\frac{\mathrm{d}\alpha_u}{\mathrm{d}\tau}=-\frac{\gamma}{\nu_u}[\nu\cos(\alpha_u+\alpha)-\nu_u]\theta_{\hat{\phi}}(n)^{\hat{r}}+\frac{\gamma}{\nu_u}(\sin\alpha_u a(n)^{\hat{r}}+\nu\nu_u\sin\alpha\, k_{(\mathrm{Lie})})-\frac{1}{m\gamma_u\nu_u}(F^1_{(\mathrm{spin})}+F^1_{(\mathrm{quad})}),$$
$$\frac{\mathrm{d}\nu_u}{\mathrm{d}\tau}=-\frac{\gamma}{\gamma_u^2}[\cos\alpha_u a(n)^{\hat{r}}+\nu\sin(\alpha_u+\alpha)\theta_{\hat{\phi}}(n)^{\hat{r}}]+\frac{1}{m\gamma_u^2}(F^2_{(\mathrm{spin})}+F^2_{(\mathrm{quad})}),$$
$$\frac{\mathrm{d}s}{\mathrm{d}\tau}=0, \quad (4.1)$$

together with the compatibility conditions,

$$0 = e_3 \times (\mathcal{E}_{(\mathrm{spin})}(u) + \mathcal{E}_{(\mathrm{quad})}(u)) + \frac{s}{m}(F_{(\mathrm{spin})} + F_{(\mathrm{quad})}), \tag{4.2}$$

which come from the spin evolution equations and yield two algebraic relations for ν and α. The last equation (4.1) implies that the signed spin magnitude s is a constant of motion. Finally, Eqs. (4.1) must be coupled with the decomposition (3.9) and (3.19) of $U = \mathrm{d}x^\alpha/\mathrm{d}\tau$ to provide the remaining unknowns t, r and ϕ [see also Eqs. (3.10)]. Equations (4.1) and (3.10) are written in a form that is suitable for the numerical integration. Additional (nonindependent) relations are obtained from the conservation (2.15) of the total energy and angular momentum,

$$E = N\gamma_u[m + s(\nu_u \sin\alpha_u a(n)^{\hat{r}} + \theta_{\hat{\phi}}(n)^{\hat{r}})] - N^\phi J,$$
$$J = \gamma_u \sqrt{g_{\phi\phi}}[m\nu_u \sin\alpha_u - s(k_{(\mathrm{Lie})} + \nu_u \sin\alpha_u \theta_{\hat{\phi}}(n)^{\hat{r}})]. \tag{4.3}$$

These can be used as a consistency check. Examples of numerically integrated orbits are discussed in Refs. [42,43].

We are interested here in studying the general features of equatorial motion to second order in spin, taking advantage of the simplified form of the MPD equations discussed in Sec. II B. We shall also derive analytic solutions for the orbits to that order by computing the corrections, produced by a nonzero spin, to a reference circular geodesic motion.

B. Solution to the order $\mathcal{O}(S^2)$

Let us introduce the dimensionless spin parameter

$$\hat{s} = \frac{s}{m_0 M}, \tag{4.4}$$

where m_0 denotes the "bare" mass of the body. Hereafter, we shall systematically neglect terms that are of orders higher than the second in $\hat{s}$. Hence, all quantities must be understood as being evaluated up to the order $\mathcal{O}(\hat{s}^2)$. Our set of equations can then be simplified by means of Eqs. (2.13) and (2.14), which yield

$$m = m_0 - \frac{1}{4}C_Q m_0 M^2 \hat{s}^2 \gamma_u^2[(2E_{\hat{r}\hat{r}} + E_{\hat{\theta}\hat{\theta}})\nu_u^2 \cos 2\alpha_u + (2+\nu_u^2)E_{\hat{\theta}\hat{\theta}} + 4\nu_u \sin\alpha_u H_{\hat{r}\hat{\theta}}], \tag{4.5}$$

and

$$\nu = \nu_u + \frac{1}{2}(1-C_Q)M^2\hat{s}^2\{\nu_u[(2E_{\hat{r}\hat{r}} + E_{\hat{\theta}\hat{\theta}})\cos 2\alpha_u + 3E_{\hat{\theta}\hat{\theta}}] + 2H_{\hat{r}\hat{\theta}}(1+\nu_u^2)\sin\alpha_u\},$$
$$\alpha = \alpha_u - (1-C_Q)M^2\hat{s}^2\frac{\cos\alpha_u}{\nu_u}[(2E_{\hat{r}\hat{r}} + E_{\hat{\theta}\hat{\theta}})\nu_u \sin\alpha_u - H_{\hat{r}\hat{\theta}}], \tag{4.6}$$

respectively. Here, $E_{\hat{r}\hat{r}}$, $E_{\hat{\theta}\hat{\theta}}$ are components of the electric part of the Riemann tensor, while $H_{\hat{r}\hat{\theta}}$ is a component of its magnetic part (see Appendix C). Equations (4.6) show that the value of the polarizability for black holes, namely $C_Q = 1$, is a very special case leading to a great simplification (see below).

On the other hand, solving Eqs. (4.3) algebraically for ν_u and α_u leads to

$$\gamma_u = \frac{E + N^\phi J}{m_0 N}\left\{1 - M\hat{s}\left[\theta_{\hat{\phi}}(n)^{\hat{r}} + \frac{Na(n)^{\hat{r}}J}{(E+N^\phi J)\sqrt{g_{\phi\phi}}}\right] - M^2\hat{s}^2\left(\frac{m_2}{m_0 M^2} + E_{\hat{r}\hat{r}} + E_{\hat{\theta}\hat{\theta}}\right)\right\},$$
$$\gamma_u \nu_u \sin\alpha_u = \frac{J}{m_0\sqrt{g_{\phi\phi}}}\left\{1 + M\hat{s}\left[\theta_{\hat{\phi}}(n)^{\hat{r}} + (E + N^\phi J)\frac{\sqrt{g_{\phi\phi}}k_{(\mathrm{Lie})}}{NJ}\right] - M^2\hat{s}^2\left(\frac{m_2}{m_0 M^2} + E_{\hat{r}\hat{r}} + E_{\hat{\theta}\hat{\theta}}\right)\right\}. \tag{4.7}$$

Those identities can be used next at the lowest order in the previous equations so as to express m, ν and α in terms of E and J. In particular, the solution for the body mass is found to be

$$m = m_0\left\{1 - \frac{1}{2}C_Q\frac{M^3}{r^3}\left[1 + \frac{3}{m_0^2 r^2}(J - aE)^2\right]\hat{s}^2\right\}$$
$$\equiv m_0 + \hat{s}^2 m_2. \tag{4.8}$$

Its behavior as a function of the radial coordinate is shown in Fig. 1 for selected values of the parameters. It is interesting to evaluate the difference between the limiting values at the horizon and at infinity, $|m(+\infty) - m(r_+)| = \hat{s}^2 m_2(r_+)$, since this represents the largest mass variation during the evolution.

It can be verified that the solution (4.6) for U is equivalently obtained from the compatibility conditions (4.2). Furthermore, the truncated equation (4.1) for the mass reads

$$\frac{\mathrm{d}m}{\mathrm{d}\tau} = F^0_{(\mathrm{quad})} + \mathcal{O}(\hat{s}^3), \tag{4.9}$$

with

$$F^0_{(\mathrm{quad})} = \frac{1}{2}m_0 M^2\hat{s}^2 C_Q \gamma_u \nu_u \cos\alpha_u\left\{\gamma_u^2\nu_u[\nu_u(b_1\cos 2\alpha_u + c_3) + 2(a_1 - b_4)\sin\alpha_u] + \frac{1}{3}(2c_1 - c_2)\right\}, \tag{4.10}$$

where the lowest-order piece of the solution (4.7) may be used, which implies notably that $F^0_{(\mathrm{spin})} = \mathcal{O}(\hat{s}^3)$.

The quantities a_i, b_i and c_i, all functions of r, are listed in Appendix D. Once parametrized with the radial coordinate r instead of the proper time τ by means of Eqs. (3.10), formula (4.9) takes the very simple form

$$\frac{\mathrm{d}m_2}{\mathrm{d}r} = \frac{3}{2} m_0 C_Q \frac{M^3}{r^4}\left[1 + \frac{5}{m_0^2 r^2}(J - aE)^2\right], \tag{4.11}$$

whose solution coincides with that of Eq. (4.8).

C. Circular motion

1. Solution to the order $\mathcal{O}(\hat{s}^2)$

In this subsection, we restrict ourselves to circular orbits, as described by the parametrizations (3.11)–(3.13). For circular motion in the equatorial plane, we must set $\alpha = \pi/2$, so that Eqs. (4.6) become

$$\begin{aligned} \nu &= \nu_u + (1 - C_Q) M^2 \hat{s}^2 [-\nu_\pm (E_{\hat{r}\hat{r}} - E_{\hat{\theta}\hat{\theta}}) + H_{\hat{r}\hat{\theta}}(1 + \nu_\pm^2)], \\ \alpha_u &= \frac{\pi}{2}, \end{aligned} \tag{4.12}$$

to second order in $\hat{s}$. Thus, $F^0_{(\mathrm{quad})} = 0$ from its expression (4.10), and Eq. (4.9) tells us that the mass of the body is a constant of motion to that order. However, it differs from the bare mass m_0 by virtue of the general definition (4.5), which yields

$$\begin{aligned} m &= m_0 - \frac{1}{2} C_Q m_0 M^2 \hat{s}^2 \gamma_\pm^2 [E_{\hat{\theta}\hat{\theta}} - \nu_\pm^2 E_{\hat{r}\hat{r}} + 2\nu_\pm H_{\hat{r}\hat{\theta}}] \\ &= m_0 \left[1 - \frac{1}{2} C_Q \frac{M^3}{r^{7/2}} \frac{r^2 \mp 4a\sqrt{Mr} + 3a^2}{r^{3/2} - 3M\sqrt{r} \pm 2a\sqrt{M}} \hat{s}^2\right], \end{aligned} \tag{4.13}$$

once evaluated at $\alpha_u = \pi/2$, with ν_u replaced by $\nu_\pm$ to the lowest order.

Next, solving algebraically the second equation (4.1) for ν_u we find

$$\nu_u = \nu_\pm + \hat{s}\nu_{u(1)} + \hat{s}^2 \nu_{u(2)}, \tag{4.14}$$

where $\nu_\pm$ denote the geodesic linear velocities (3.14), while $\nu_{u(1)}$ and $\nu_{u(2)}$ are the spin-induced corrections

$$\begin{aligned} \nu_{u(1)} &= \pm \frac{M}{2\zeta_K} [-\nu_\pm (E_{\hat{r}\hat{r}} - E_{\hat{\theta}\hat{\theta}}) + (1 + \nu_\pm^2) H_{\hat{r}\hat{\theta}}], \\ \nu_{u(2)} &= \mp \frac{M^2}{\gamma_\pm^2 \zeta_K} \tilde{F}^1_{(\mathrm{quad})} + \nu_{u(1)} \left[\nu_{u(1)}\left(\pm \frac{k_{(\mathrm{Lie})}}{2\zeta_K} + \frac{1}{\nu_\pm}\right) \right. \\ &\quad \left. \pm (1 - C_Q) M \zeta_K \pm \frac{M H_{\hat{r}\hat{\theta}}}{2\gamma_\pm^2 \nu_\pm \zeta_K}\right], \end{aligned} \tag{4.15}$$

of first and second orders, respectively, with [see also Eq. (E5)]

$$\begin{aligned} \tilde{F}^1_{(\mathrm{quad})} = \frac{1}{4} C_Q \bigg\{ & \gamma_\pm^2 [b_1 - b_2 + \nu_\pm (b_4 - b_5)] \\ & - b_1 + \frac{1}{3}(b_2 + b_3) \bigg\}. \end{aligned} \tag{4.16}$$

Substituting the above solutions for ν_u and α_u into Eq. (4.3), we obtain

$$\begin{aligned} E &= m_0 N \gamma_\pm \left(1 - \frac{\sqrt{g_{\phi\phi}} N^\phi}{N} \nu_\pm\right) + m_0 \gamma_\pm [\gamma_\pm^2 \zeta_\pm \nu_{u(1)} \sqrt{g_{\phi\phi}} + MN(\nu_\pm a(n)^{\hat{r}} + \theta_{\hat{\phi}}(n)^{\hat{r}}) + M\sqrt{g_{\phi\phi}} N^\phi (\nu_\pm \theta_{\hat{\phi}}(n)^{\hat{r}} + k_{(\mathrm{Lie})})]\hat{s} \\ &\quad + \left\{ m_2 N \gamma_\pm \left(1 - \frac{\sqrt{g_{\phi\phi}} N^\phi}{N} \nu_\pm\right) + m_0 \gamma_\pm^3 \left[\sqrt{g_{\phi\phi}} \zeta_\pm \left(\pm M \zeta_K \nu_{u(1)} + \nu_{u(2)} + \frac{3}{2} \gamma_\pm^2 \nu_\pm \nu_{u(1)}^2\right) + \frac{1}{2} N \nu_{u(1)}^2\right]\right\} \hat{s}^2, \\ J &= m_0 \gamma_\pm \nu_\pm \sqrt{g_{\phi\phi}} + m_0 \gamma_\pm \sqrt{g_{\phi\phi}} [\gamma_\pm^2 \nu_{u(1)} - M(\nu_\pm \theta_{\hat{\phi}}(n)^{\hat{r}} + k_{(\mathrm{Lie})})]\hat{s} \\ &\quad + \left[m_2 \gamma_\pm \nu_\pm \sqrt{g_{\phi\phi}} + m_0 \gamma_\pm^3 \sqrt{g_{\phi\phi}} \left(\pm M \zeta_K \nu_{u(1)} + \nu_{u(2)} + \frac{3}{2} \gamma_\pm^2 \nu_\pm \nu_{u(1)}^2\right)\right] \hat{s}^2. \end{aligned} \tag{4.17}$$

The behavior of the energy E versus the angular momentum J in the case of corotating orbits is shown in Fig. 2 for selected values of the parameters.

Finally, the 4-velocity U is given by Eq. (3.11), with normalization factor

$$\Gamma = \Gamma_\pm \left\{1 + \gamma_\pm^2 \nu_\pm \nu_1 \hat{s} + \gamma_\pm^2 \left[\nu_\pm \nu_2 + \left(\frac{3}{2}\gamma_\pm^2 - 1\right)\nu_1^2\right]\hat{s}^2\right\}, \tag{4.18}$$

and angular velocity

$$\zeta = \zeta_\pm \left[1 + \frac{N}{\zeta_\pm \sqrt{g_{\phi\phi}}}(\nu_1 \hat{s} + \nu_2 \hat{s}^2)\right], \tag{4.19}$$

where

$$\nu_1 = \nu_{u(1)}, \qquad \nu_2 = \nu_{u(2)} \pm 2(1 - C_Q) M \zeta_K \nu_1. \tag{4.20}$$

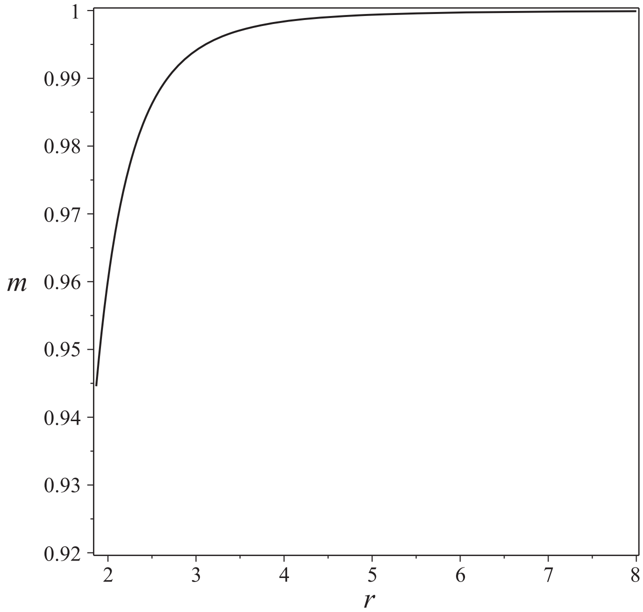

FIG. 1. The behavior of the mass m of the body (in units of m_0) as a function of the radial coordinate (in units of M) is shown for the following choice of parameters: $a/M = 0.5$, $C_Q = 1$, $\hat{s} = 0.25$, $E/m_0 = 1$ and $J/(m_0 M) = 4$. The mass shift in this case equates $|m(+\infty) - m(r_+)| \approx 0.055 m_0$. The value of the dimensionless spin parameter $\hat{s}$ has been exaggerated in order to enhance the effect.

The relation between the timelike unit tangent vector U to the body center world line and the unit timelike vector u aligned with the 4-momentum, thus, reads

$$U - u = \hat{s}^2 \gamma_\pm^2 (1 - C_Q) M \zeta_K \nu_1 \bar{U}_\pm, \tag{4.21}$$

the unit vector $\bar{U}_\pm$ being already defined in Eq. (3.16). Hence, in general, U and P are not aligned unless $C_Q = 1$, as discussed below.

2. *Weak-field limit*

Let us study now the weak-field limit of the above analysis. For convenience, we introduce the dimensionless quantities $u_0 = M/r_0$ and $\hat{a} = a/M$. We may consider only the case of corotating orbits, the counter-rotating case simply following from the replacement $\hat{a} \to -\hat{a}$. Every quantity is expanded up to a certain power of u_0 as follows,

$$X \sim X_0 + \hat{a} X_{\hat{a}} + \hat{s} X_{\hat{s}} + \hat{a}^2 X_{\hat{a}\hat{a}} + 2\hat{a}\hat{s} X_{\hat{a}\hat{s}} + \hat{s}^2 X_{\hat{s}\hat{s}}, \tag{4.22}$$

where terms of orders higher than the second in the background rotation parameter, as well as terms like $\hat{a}\hat{s}^2$ and $\hat{a}^2\hat{s}$, are neglected.

The weak-field expansion of the conserved energy and angular momentum (4.17) are then found to be

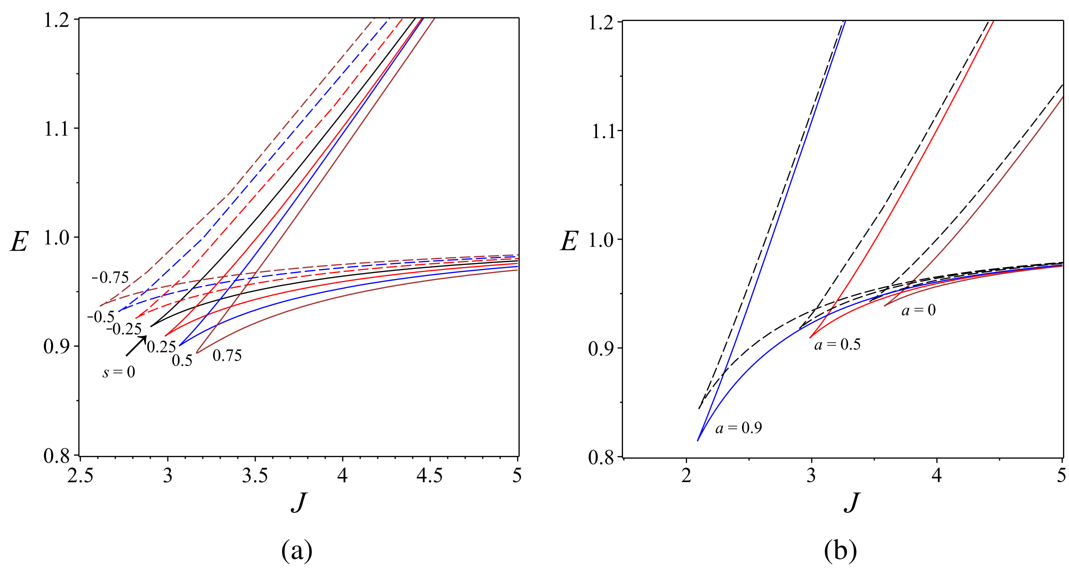

FIG. 2 (color online). The behavior of the energy E versus the angular momentum J in the case of corotating circular orbits is shown in panel (a), for the fixed black-hole dimensionless spin parameter $a/M = 0.5$ and different values of $\hat{s}$; in panel (b), for the fixed body spin $\hat{s} = 0.25$ and different values of a/M; $C_Q = 1$ in both cases. Dashed curves in (b) correspond to the geodesic motion ($\hat{s} = 0$).

$$\frac{E}{m_0} = 1 - \frac{1}{2}u_0 + \frac{3}{8}u_0^2 + \frac{27}{16}u_0^3 + \frac{675}{128}u_0^4 - \frac{1}{16}u_0^{5/2}(8 + 36u_0 + 135u_0^2)(2\hat{a} + \hat{s})$$
$$+ \frac{1}{4}u_0^3(2 + 15u_0)\hat{a}^2 + \frac{1}{4}u_0^3(2 + 23u_0)\hat{a}\hat{s} + \frac{1}{8}u_0^3[2C_Q - (6 - 17C_Q)u_0]\hat{s}^2 + \mathcal{O}(u_0^5),$$
$$\frac{J}{m_0 M} = u_0^{-1/2}\left(1 + \frac{3}{2}u_0 + \frac{27}{8}u_0^2 + \frac{135}{16}u_0^3 + \frac{2835}{128}u_0^4 + \frac{15309}{256}u_0^5\right) - 3u_0\left(1 + \frac{5}{2}u_0 + \frac{63}{8}u_0^2 + \frac{405}{16}u_0^3\right)\hat{a}$$
$$+ \left(1 - 2u_0 - \frac{27}{8}u_0^2 - \frac{81}{8}u_0^3 - \frac{4185}{128}u_0^4\right)\hat{s} + u_0^{3/2}\left(1 + 5u_0 + \frac{189}{8}u_0^2 + \frac{405}{4}u_0^3\right)\hat{a}^2$$
$$+ \frac{3}{2}u_0^{3/2}\left(1 + \frac{9}{2}u_0 + \frac{159}{8}u_0^2 + \frac{1305}{16}u_0^3\right)\hat{a}\hat{s}$$
$$+ \frac{3}{4}u_0^{3/2}\left[C_Q - \frac{1}{2}\left(5 - \frac{23}{3}C_Q\right)u_0 - \frac{1}{4}\left(7 - \frac{79}{2}C_Q\right)u_0^2 + \frac{1}{16}(81 + 477C_Q)u_0^3\right]\hat{s}^2 + \mathcal{O}(u_0^5), \tag{4.23}$$

whereas the normalization factor (4.18) and the angular velocity (4.19) become

$$\Gamma = 1 + \frac{3}{2}u_0 + \frac{27}{8}u_0^2 + \frac{135}{16}u_0^3 + \frac{2835}{128}u_0^4 - \frac{3}{16}u_0^{5/2}(8 + 36u_0 + 135u_0^2)(2\hat{a} + \hat{s})$$
$$+ \frac{1}{4}u_0^3(2 + 27u_0)\hat{a}^2 + \frac{3}{4}u_0^3(2 + 19u_0)\hat{a}\hat{s} + \frac{3}{8}u_0^3[2C_Q + 5(2 + C_Q)u_0]\hat{s}^2 + \mathcal{O}(u_0^5),$$
$$M\zeta = u_0^{3/2} - \left(\hat{a} + \frac{3}{2}\hat{s}\right)u_0^3 + u_0^{9/2}\hat{a}^2 + \frac{3}{2}u_0^{7/2}(1 + 2u_0)\hat{a}\hat{s} + \frac{3}{4}u_0^{7/2}\left[C_Q + \left(\frac{7}{2} - 2C_Q\right)\right]\hat{s}^2 + \mathcal{O}(u_0^5), \tag{4.24}$$

respectively. In order to derive gauge-invariant expressions, we express E and J in terms of the gauge-invariant dimensionless variable $y = (M\zeta)^{2/3}$, related to u_0 by

$$u_0 = y + \left(\frac{2}{3}\hat{a} + \hat{s}\right)y^{5/2} + \frac{5}{9}y^4\hat{a}^2 - y^3\left(1 - \frac{5}{3}y\right)\hat{a}\hat{s} - \frac{1}{2}y^3[C_Q - 2(1 + C_Q)y]\hat{s}^2 + \mathcal{O}(y^{9/2}). \tag{4.25}$$

This yields

$$\frac{E}{m_0} = 1 - \frac{1}{2}y + \frac{3}{8}y^2 + \frac{27}{16}y^3 + \frac{675}{128}y^4 - y^{5/2}\left(\frac{4}{3} + 4y + \frac{27}{2}y^2\right)\hat{a} - y^{5/2}\left(1 + \frac{3}{2}y + \frac{27}{8}y^2\right)\hat{s}$$
$$+ \frac{1}{2}y^3\left(1 + \frac{65}{18}y\right)\hat{a}^2 + y^3\left(1 + \frac{5}{6}y\right)\hat{a}\hat{s} + \frac{1}{2}y^3\left[C_Q - 5\left(1 - \frac{1}{2}C_Q\right)y\right]\hat{s}^2 + \mathcal{O}(y^5),$$
$$\frac{J}{m_0 M} = y^{-1/2}\left(1 + \frac{3}{2}y + \frac{27}{8}y^2 + \frac{135}{16}y^3 + \frac{2835}{128}y^4 + \frac{15309}{256}y^5\right) - y\left(\frac{10}{3} + 7y + \frac{81}{4}y^2 - \frac{495}{8}y^3\right)\hat{a}$$
$$+ \left(1 - \frac{5}{2}y - \frac{21}{8}y^2 - \frac{81}{16}y^3 - \frac{1485}{128}y^4\right)\hat{s} + y^{3/2}\left(1 + \frac{26}{9}y + \frac{335}{24}y^2 + \frac{459}{8}y^3\right)\hat{a}^2$$
$$+ y^{3/2}\left(2 + \frac{4}{3}y + \frac{25}{4}y^2 + \frac{81}{4}y^3\right)\hat{a}\hat{s}$$
$$+ y^{3/2}\left[C_Q - 4\left(1 - \frac{1}{2}C_Q\right)y - \frac{15}{2}\left(1 - \frac{3}{4}C_Q\right)y^2 - \frac{81}{4}\left(1 - \frac{5}{6}C_Q\right)y^3\right]\hat{s}^2 + \mathcal{O}(y^5). \tag{4.26}$$

3. *Comparison with Ref. [45]*

Before investigating the general equatorial motion, let us show how the above analysis allows us to reproduce the results of Ref. [45]. As already stated, our quadrupole tensor reduces to the one adopted there to describe black holes if we set $C_Q = 1$. Since we have then $\nu = \nu_u + \mathcal{O}(\hat{s}^3)$ from Eq. (4.12) [see also Eq. (4.21)], this entails $P \propto U + \mathcal{O}(\hat{s}^3)$. In order to write our expressions for the energy and angular momentum in the same form as in Ref. [45], we eliminate the dependence on the radial coordinate in favor of the angular velocity by inverting perturbatively Eq. (4.19):

$$r = r_c + \frac{M}{r_c}\left(-\sqrt{Mr_c} \pm a\right)\hat{s} + \frac{M^2}{2r_c^3}\left\{\Delta_c - 2a\left(a \mp \sqrt{Mr_c}\right) - (1 - C_Q)\left[\Delta_c + 4a\left(a \mp \sqrt{Mr_c}\right)\right]\right\}\hat{s}^2, \tag{4.27}$$

where $r_c^{3/2} = \sqrt{M}(\frac{1}{\zeta} \mp a)$ and $\Delta_c = \Delta(r = r_c)$. Substituting next into Eq. (4.17) leads to

$$\begin{aligned}
E &= \frac{m_0}{r_c^{3/4}}\frac{r_c^{3/2} - 2M\sqrt{r_c} \pm a\sqrt{M}}{[r_c^{3/2} - 3M\sqrt{r_c} \pm 2a\sqrt{M}]^{1/2}} \mp \frac{m_0 M^2}{r_c^{9/4}}\frac{\sqrt{Mr_c} \mp a}{[r_c^{3/2} - 3M\sqrt{r_c} \pm 2a\sqrt{M}]^{1/2}}\hat{s} \\
&\quad + \frac{m_0 M^3}{2r_c^{15/4}}\frac{1}{[r_c^{3/2} - 3M\sqrt{r_c} \pm 2a\sqrt{M}]^{1/2}}\Bigg\{r_c^{3/2} - 4M\sqrt{r_c} \pm 3a\sqrt{M} \\
&\quad - (1 - C_Q)\left[r_c^{3/2} + M\sqrt{r_c} \mp 9a\sqrt{M} + \frac{11a^2}{2\sqrt{r_c}} + \frac{3}{2}\frac{(r_c + 3M)a^2 + 2M^{3/2}\sqrt{r_c}(\sqrt{Mr_c} \mp 3a)}{r_c^{3/2} - 3M\sqrt{r_c} \pm 2a\sqrt{M}}\right]\Bigg\}\hat{s}^2, \\
J &= \pm\frac{m_0\sqrt{M}}{r_c^{3/4}}\frac{r_c^2 \mp 2a\sqrt{Mr_c} + a^2}{[r_c^{3/2} - 3M\sqrt{r_c} \pm 2a\sqrt{M}]^{1/2}} + \frac{m_0 M}{r_c^{9/4}}\frac{r_c^2(r_c - 4M) + Ma^2 \pm a\sqrt{Mr_c}(3r_c - M)}{[r_c^{3/2} - 3M\sqrt{r_c} \pm 2a\sqrt{M}]^{1/2}}\hat{s} \\
&\quad \pm \frac{m_0 M^{5/2}}{2r_c^{15/4}}\frac{1}{[r_c^{3/2} - 3M\sqrt{r_c} \pm 2a\sqrt{M}]^{1/2}}\Bigg\{3Ma^2 \pm 2a\sqrt{Mr_c}(3r_c - 2M) + r_c^2(2r_c - 7M) \\
&\quad - (1 - C_Q)\Bigg[(2r_c + 3M)(r_c^2 - Mr_c + 3M^2) \mp 4a\sqrt{Mr_c}(3r_c + 2M) + 10r_c a^2 \pm \frac{11\sqrt{M}a^3}{2\sqrt{r_c}} \\
&\quad + \frac{3}{2}\sqrt{M}\frac{6M^{3/2}\sqrt{r_c}(3M^2 + 4a^2) \mp a[a^2(r_c + 9M) + 4M^2(5r_c + 3M)]}{r_c^{3/2} - 3M\sqrt{r_c} \pm 2a\sqrt{M}}\Bigg]\Bigg\}\hat{s}^2.
\end{aligned} \tag{4.28}$$

This exactly reproduces the results of Ref. [45] when specialized to the case $C_Q = 1$ [see their counterparts displayed in Eqs. (39a) and (39b) with, in addition, $M = 1 = m_0$].

D. General equatorial motion

1. Effective potential

In order to discuss the general features of equatorial motion, it is most useful to introduce appropriate effective potentials [56,57]. The latter naturally arise when factoring the expression of $(\mathrm{d}r/\mathrm{d}\tau)^2$ as a polynomial in the energy E of the test object. This factorisation takes the form

$$\begin{aligned}
\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 &= \frac{\gamma^2\nu^2\cos^2\alpha}{g_{rr}} \\
&= \mathcal{A}\mathcal{P}(E)(E - V_{(+)})(E - V_{(-)}) + \mathcal{O}(\hat{s}^3),
\end{aligned} \tag{4.29}$$

where the solutions for ν and α in terms of the conserved energy and angular momentum are given by Eqs. (4.6)–(4.7). More precisely, we find

$$\begin{aligned}
\mathcal{A} &= \frac{1}{m_0^2 N^2 g_{rr}}\left\{1 - 2M\hat{s}\theta_{\hat{\phi}}(n)^{\hat{r}} - M^2\hat{s}^2\left[k_{(\mathrm{Lie})}(k_{(\mathrm{Lie})} - a(n)^{\hat{r}}) + (2 - C_Q)(2E_{\hat{r}\hat{r}} + E_{\hat{\theta}\hat{\theta}})\frac{J^2}{m_0^2 g_{\phi\phi}} + (3 - C_Q)E_{\hat{\theta}\hat{\theta}} + 3E_{\hat{r}\hat{r}}\right]\right\}, \\
\mathcal{P}(E) &= 1 + \frac{M^2\hat{s}^2}{m_0^2 N^2}(2 - C_Q)(E + N^\phi J)\left[(E_{\hat{r}\hat{r}} + 2E_{\hat{\theta}\hat{\theta}})(E + N^\phi J) + 2\frac{N}{\sqrt{g_{\phi\phi}}}H_{\hat{r}\hat{\theta}}J\right],
\end{aligned} \tag{4.30}$$

and

$$V_{(\pm)} = V_0^{(\pm)} + \hat{s}V_1^{(\pm)} + \hat{s}^2 V_2^{(\pm)}, \tag{4.31}$$

with

$$V_0^{(\pm)} = -N^\phi J \pm \frac{N}{\sqrt{g_{\phi\phi}}}\sqrt{m_0^2 g_{\phi\phi} + J^2},$$
$$V_1^{(\pm)} = \frac{MN}{\sqrt{g_{\phi\phi}}}\left[(a(n)^{\hat r} + k_{(\mathrm{Lie})})J \pm \theta_{\hat\phi}(n)^{\hat r}\frac{m_0^2 g_{\phi\phi} + 2J^2}{\sqrt{m_0^2 g_{\phi\phi} + J^2}}\right],$$
$$V_2^{(\pm)} = \pm\frac{M^2 N}{2\sqrt{g_{\phi\phi}}}\left\{\pm 2[\theta_{\hat\phi}(n)^{\hat r}(a(n)^{\hat r} + 3k_{(\mathrm{Lie})}) - C_Q H_{\hat r\hat\theta}]J + \sqrt{m_0^2 g_{\phi\phi} + J^2}[2k_{(\mathrm{Lie})}(k_{(\mathrm{Lie})} + 3a(n)^{\hat r}) - (4 - C_Q)E_{\hat r\hat r} - (4 + C_Q)E_{\hat\theta\hat\theta}]\right\}$$
$$\mp\frac{M^2 m_0^2 N\sqrt{g_{\phi\phi}}}{2\sqrt{m_0^2 g_{\phi\phi} + J^2}}\left\{[k_{(\mathrm{Lie})}(k_{(\mathrm{Lie})} + 3a(n)^{\hat r}) - (3 - C_Q)E_{\hat r\hat r} - 3E_{\hat\theta\hat\theta}] + \frac{m_0^2 g_{\phi\phi}}{m_0^2 g_{\phi\phi} + J^2}[k_{(\mathrm{Lie})}a(n)^{\hat r} - (E_{\hat r\hat r} + E_{\hat\theta\hat\theta})]\right\}. \tag{4.32}$$

If E and J are kept fixed as $\hat s$ goes to zero, the leading order approximation of $\mathcal{P}(E)$ is obtained by setting exactly $\hat s = 0$ in Eq. (4.30), which shows that $\mathcal{P}(E)$ is necessarily positive in the domain of validity of the small spin expansion. The solutions $E = V_{(\pm)}$ are generalizations of the radial effective potentials for a test particle in a Kerr spacetime to the case of an extended body with spin-induced quadrupole moment. Their behavior as a function of the radial distance is shown in Fig. 3 for selected values of the parameters. The upper/lower branch corresponds to the $+/-$ sign in Eq. (4.32).

On the other hand, since the equation $(\mathrm{d}r/\mathrm{d}\tau)^2 = 0$ is quartic in E, in order to give a complete account of effective potentials we should also consider the solutions to the equation $\mathcal{P}(E) = 0$ or, equivalently,

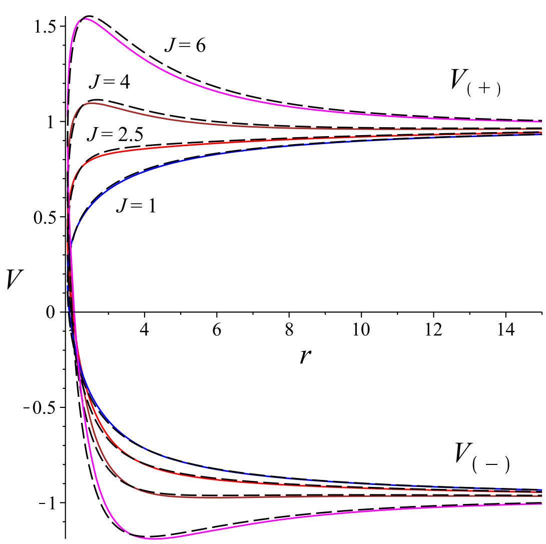

FIG. 3 (color online). The behavior of the effective potential for radial motion is shown for the following choice of parameters: $a/M = 0.5$, $C_Q = 1$, $\hat s = 0.25$, and for different values of the body dimensionless angular momentum, $J/(m_0 M) = [1, 2.5, 4, 6]$. The corresponding geodesic case ($\hat s = 0$) is also shown for comparison (dashed curves).

$$1 + \frac{3aM^3\hat s^2}{m_0^2 r^5}(2 - C_Q)(E + N^\phi J) \times \left[a(E + N^\phi J) - 2\frac{N^2}{\Delta}(r^2 + a^2)J\right] = 0. \tag{4.33}$$

If $C_Q < 2$, no real solutions for the energy exist. If $C_Q = 2$, then $\mathcal{P}(E) = 1$, irrespective of E and J. If $C_Q > 2$ (so excluding the black hole case $C_Q = 1$), the above equation admits two real solutions $E = W_{(\pm)}$, perturbatively in $\hat s$, only if E scales as $1/\hat s$, with

$$W_{(\pm)} = \frac{m_0}{\hat a}\left[\pm\frac{1}{\hat s\sqrt{3(C_Q - 2)}}\left(\frac{r}{M}\right)^{5/2} + \frac{J}{m_0 M}\right] + \mathcal{O}(\hat s). \tag{4.34}$$

Note that the solutions $W_{(\pm)}$ diverge in both limits $\hat a \to 0$ and $\hat s \to 0$ for fixed values of the radial distance. Furthermore, for fixed values of the dimensionless spin parameters, they indefinitely grow for large r, exhibiting a monotonic behavior $W_{(\pm)} \sim \pm(r/M)^{5/2}$, so that there cannot exist circular orbits associated with them. In the following, we shall actually exclude the configurations for which $E = \mathcal{O}(1/\hat s)$ and ignore both potentials $W_{(\pm)}$.

2. Circular orbits and ISCO

Circular orbits correspond to the extremal points of $V_{(+)}$ and solving for J the resulting equation, $V'_{(+)} = 0$, provides the associated angular momentum. Now, as pointed out by Le Tiec *et al.* [58], the shift of the ISCO frequency due to the peturbation induced to the spacetime background metric by the particle itself is an important strong-field benchmark. Gravitational self-force theory has provided very accurate analytic predictions for it in the case of a spinless body in motion along a circular geodesic on a Schwarzschild background, at the first order in the symmetric mass ratio of the two objects. In the present situation, because of its spinning and quadrupolar structure, the body deviates from geodesic motion in the way described by the MPD model. As a result, the last stable circular orbit undergoes a shift in the frequency, made of terms proportional to the spin as well as the quadrupole. Measuring this effect can therefore provide relevant information on the structure of the body. Conversely, having information about the spin and quadrupolar structure of the body allows one to make predictions on the frequency of circular motion and its deviation from the corresponding geodesic value.

Stability requires that $V''_{(+)} \leq 0$. For a spinless object the latter condition boils down to

$$r^2 - 6Mr - 3a^2 \pm 8a\sqrt{Mr} \geq 0. \tag{4.35}$$

The equality can be analytically solved for the radius $r_{\rm ms}$ of the marginally stable orbit (or ISCO) [59]:

$$r_{\rm ms}^{\rm Kerr} = M[3 + Z_2 \mp \sqrt{(3 - Z_1)(3 + Z_1 + 2Z_2)}], \tag{4.36}$$

where the upper or lower sign refers to co- or counter-rotating orbits, respectively, with

$$\begin{aligned} Z_1 &= 1 + (1 - \hat{a}^2)^{1/3}[(1 - \hat{a})^{1/3} + (1 + \hat{a})^{1/3}], \\ Z_2 &= \sqrt{3\hat{a}^2 + Z_1^2}. \end{aligned} \tag{4.37}$$

The latter quantities are even functions of $\hat{a}$ satisfying $Z_2 \geq Z_1$, $Z_1 \leq 3$ and, for $\hat{a} = 0$, $Z_1 = 3 = Z_2$. For small values of $\hat{a}$, the Kerr ISCO radius may be expanded as

$$\frac{r_{\rm ms}^{\rm Kerr}}{M} = 6 \mp \frac{4\sqrt{6}}{3}\hat{a} - \frac{7}{18}\hat{a}^2 \mp \frac{13\sqrt{6}}{162}\hat{a}^3 - \frac{241}{1944}\hat{a}^4 + \mathcal{O}(\hat{a}^5). \tag{4.38}$$

We recover the value $r_{\rm ms}^{\rm Kerr} = 6M$ in the Schwarzschild case ($\hat{a} = 0$).

For a spinning particle, the ISCO is modified as follows,

$$r_{\rm ISCO} = r_{\rm ms}^{\rm Kerr} + \hat{s} r_{\rm ms(1)} + \hat{s}^2 r_{\rm ms(2)}, \tag{4.39}$$

where

$$\begin{aligned} \frac{r_{\rm ms(1)}}{M} &= \mp \frac{2\sqrt{6}}{3} + \frac{2}{9}\hat{a} \pm \frac{17\sqrt{6}}{324}\hat{a}^2 + \frac{43}{486}\hat{a}^3 + \mathcal{O}(\hat{a}^4), \\ \frac{r_{\rm ms(2)}}{M} &= -\frac{29}{72} + \frac{1}{2}C_Q \pm \frac{\sqrt{6}}{54}\left(\frac{23}{24} + C_Q\right)\hat{a} + \frac{1}{81}\left(\frac{451}{48} - 4C_Q\right)\hat{a}^2 \pm \frac{\sqrt{6}}{1296}\left(\frac{4559}{72} - 37C_Q\right)\hat{a}^3 + \mathcal{O}(\hat{a}^4). \end{aligned} \tag{4.40}$$

Of course, both the first- and second-order corrections to the ISCO can also be straightforwardly computed in the strong-field regime. However, the corresponding expressions are quite long, so we prefer not to explicitly write them down. For instance, in the corotating case, for $\hat{a} = 0.5$, we find $r_{\rm ms}^{\rm Kerr}/M \approx 4.233$, $r_{\rm ms(1)}/M \approx -1.472$ and $r_{\rm ms(2)}/M \approx 0.194$ ($C_Q = 1$), or $r_{\rm ms(2)}/M \approx 2.674$ ($C_Q = 6$). The resulting behavior of $r_{\rm ISCO}$ as a function of the spin parameter $\hat{s}$ is shown in Fig. 4.

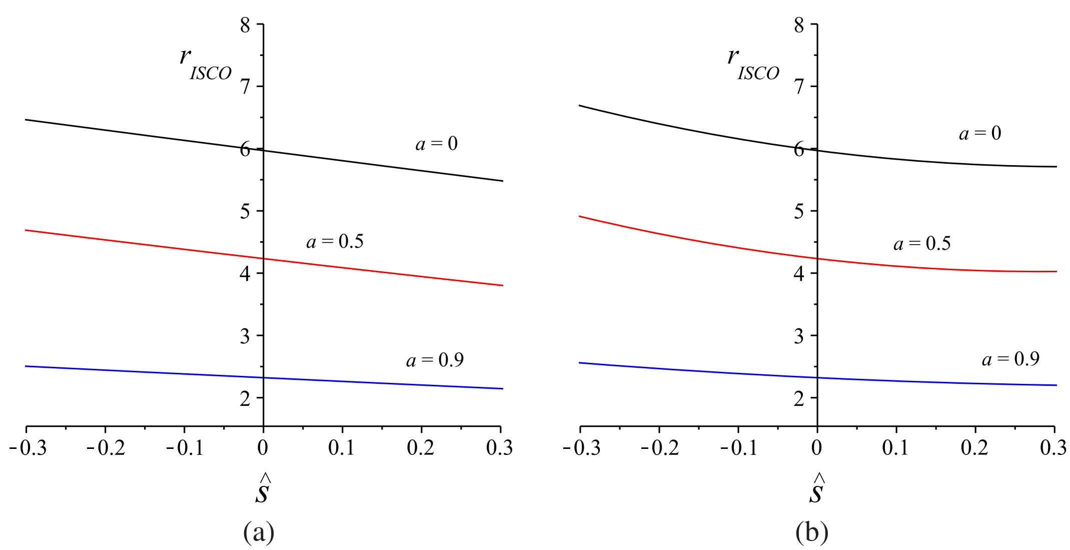

FIG. 4 (color online). The radius of the ISCO as a function of the spin parameter is shown in the case of corotating orbits for different values of the black-hole dimensionless spin, $a/M = [0, 0.5, 0.9]$, and for (a) $C_Q = 1$, as well as (b) $C_Q = 6$.

The ISCO frequency of a spinning test object is then computed to be

$$M\zeta_{\rm ISCO} = \pm 6^{-3/2}\left\{1 \pm \frac{1}{\sqrt{6}}\left(\frac{11}{6}\hat{a} + \frac{3}{4}\hat{s}\right) + \frac{59}{108}\hat{a}^2 + \frac{1}{3}\hat{a}\hat{s} + \frac{1}{9}\left(\frac{97}{64} - C_Q\right)\hat{s}^2\right\}, \tag{4.41}$$

leading to a fractional correction with respect to the spinless case,

$$\delta_{\rm ISCO} \equiv \frac{\zeta_{\rm ISCO}}{\zeta_{\rm ISCO}^{\rm Kerr}} - 1 = \pm\frac{\sqrt{6}}{8}\hat{s} + \frac{5}{48}\hat{a}\hat{s} + \frac{1}{9}\left(\frac{97}{64} - C_Q\right)\hat{s}^2, \tag{4.42}$$

whereas the energy and angular momentum at the ISCO read

$$\frac{E_{\rm ISCO}}{m_0} = \frac{2\sqrt{2}}{3} \mp \frac{\sqrt{3}}{108}(2\hat{a} + \hat{s}) - \frac{2\sqrt{2}}{3}\left[\frac{5}{216}\hat{a}^2 + \frac{1}{54}\hat{a}\hat{s} + \frac{1}{216}\left(\frac{15}{8} - C_Q\right)\hat{s}^2\right],$$
$$\frac{J_{\rm ISCO}}{m_0 M} = \pm 2\sqrt{3} - \frac{\sqrt{2}}{3}(2\hat{a} - \hat{s}) \mp 2\sqrt{3}\left[\frac{2}{27}\hat{a}^2 + \frac{11}{216}\hat{a}\hat{s} + \frac{1}{24}\left(1 - \frac{7}{9}C_Q\right)\hat{s}^2\right]. \tag{4.43}$$

The behavior of the ISCO frequency as a function of the spin parameter $\hat{s}$ is shown in Fig. 5 for two typical values of C_Q, i.e., $C_Q = 1$ (black hole) and $C_Q = 6$ (neutron star). The corresponding curve is in general a parabola, which is concave up or down depending on whether the sign of the coefficient of $\hat{s}^2$ is positive or negative. For instance, for the chosen values of the rotation parameter $\hat{a} = [0, 0.5, 0.9]$, the change of concavity (from up to down) occurs at $C_Q \approx [1.516, 1.487, 1.455]$, respectively. Finally, Fig. 6 shows the behavior of the fractional correction to the ISCO frequency as a function of the spin parameter. Furthermore, restricting $\hat{s}$ to a given range of values yields the uncertainty associated with the ISCO position, angular velocity and shift. As an example, we list the three latter quantities below in Table I for selected values of the rotational parameter $\hat{a} = [0, 0.1, 0.3, 0.5, 0.7, 0.9]$, spin parameter $\hat{s} = [-0.1, 0, 0.1]$ and polarizability parameter $C_Q = [1, 6]$, in the corotating case.

3. Quasicircular orbits

Let us finally construct the quadratic-in-spin solution to the MPD equations corresponding to a quasicircular orbit, in the perturbative sense. The initial conditions are chosen so that the world line of the extended body has the same starting point as the reference circular geodesic at radius $r = r_0$ for vanishing spin. We also require that the two world lines are initially tangent.

The orbit can be parametrized in a Keplerian-like form as follows [60,61]

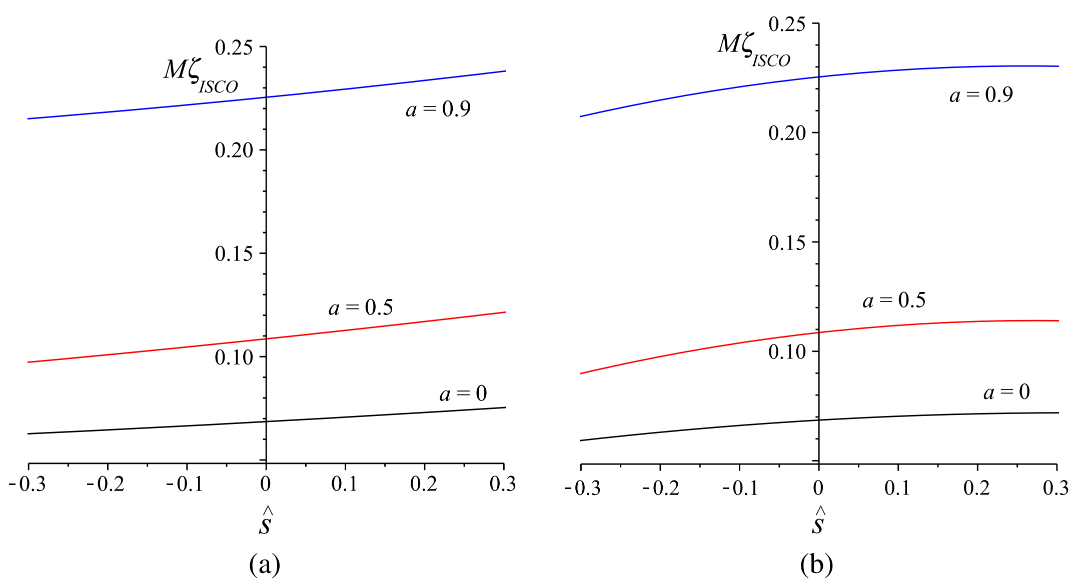

FIG. 5 (color online). The behavior of the ISCO frequency as a function of the spin parameter is shown in the case of corotating orbits for different values of the black-hole spin, $a/M = [0, 0.5, 0.9]$, and for (a) $C_Q = 1$, as well as (b) $C_Q = 6$.

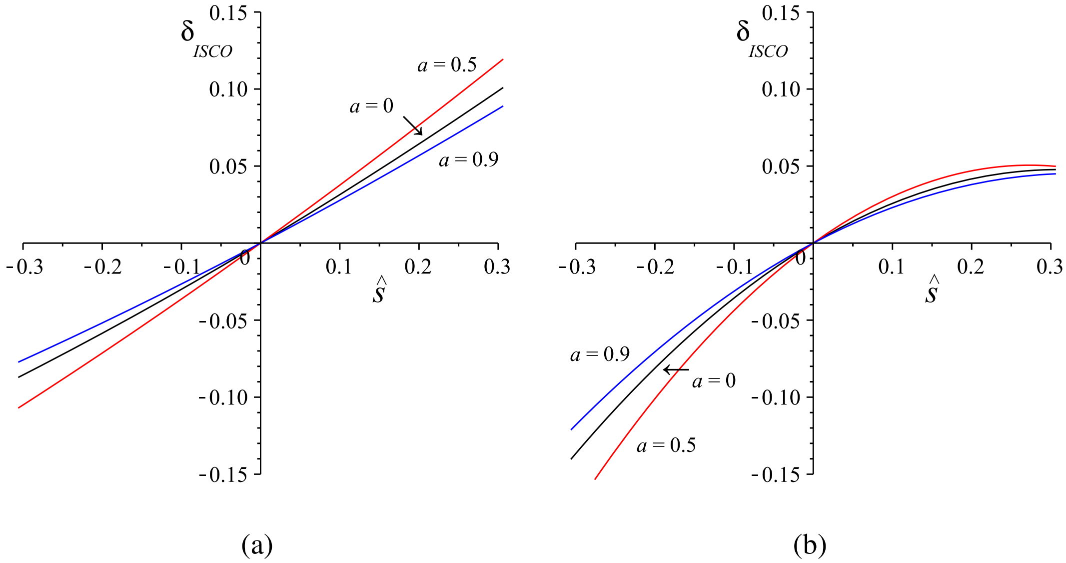

FIG. 6 (color online). The behavior of the fractional correction to the ISCO frequency as a function of the spin parameter is shown in the case of corotating orbits for different values of the black-hole dimensionless spin, $a/M = [0, 0.5, 0.9]$, and for (a) $C_Q = 1$, as well as (b) $C_Q = 6$.

$$
\begin{aligned}
\frac{2\pi}{P}(t - t_0) &= \ell_t - e_t \sin \ell_t,\\
r &= a_r(1 - e_r \cos \ell_r),\\
\theta &= \frac{\pi}{2},\\
\frac{2\pi}{\Phi}(\phi - \phi_0) &= 2 \arctan\left(\sqrt{\frac{1 + e_\phi}{1 - e_\phi}} \tan \frac{\ell_\phi}{2}\right),
\end{aligned}
\tag{4.44}
$$

where a_r is some "semimajor axis," e_t, e_r and e_ϕ are three different "eccentricities," which would coincide in the Newtonian theory, while P and Φ denote the periods of t and ϕ motions, respectively (with an abuse of notation for P, not to be confused with the body's 4-momentum). The quantities ℓ_t, ℓ_r and ℓ_ϕ are functions of the proper time parameter τ on the orbit. They are conveniently expressed in terms of the dimensionless variable $\ell = \Omega_{(\mathrm{ep})}\tau$, where

$$
\Omega_{(\mathrm{ep})} = |\Omega_\pm| \left[1 - \frac{6M}{r_0} - \frac{3a^2}{r_0^2} \pm 8a\zeta_K\right]^{1/2} \tag{4.45}
$$

denotes the well-known epicyclic frequency governing the radial perturbations of circular geodesics. The quantities ℓ_t,

TABLE I. The numerical values for the modified ISCO position, angular velocity and shift are listed for selected values of $\hat{s}$ and $\hat{a}$ in the corotating case, for $C_Q = 1$ (black hole), as well as $C_Q = 6$ (neutron star).

	$C_Q = 1$								
	r_{ISCO}/M			$M\zeta_{\mathrm{ISCO}}$			δ_{ISCO}		
$\hat{a}$	$\hat{s} = -0.1$	$\hat{s} = 0$	$\hat{s} = 0.1$	$\hat{s} = -0.1$	$\hat{s} = 0$	$\hat{s} = 0.1$	$\hat{s} = -0.1$	$\hat{s} = 0$	$\hat{s} = 0.1$
0	6.1643	6	5.8377	0.0660	0.0680	0.0702	−0.0300	0	0.0312
0.1	5.8313	5.6693	5.5094	0.0712	0.0735	0.0759	−0.0311	0	0.0323
0.3	5.1351	4.9786	4.8248	0.0847	0.0877	0.0907	−0.0335	0	0.0348
0.5	4.3824	4.2330	4.0876	0.1046	0.1086	0.1127	−0.0365	0	0.0376
0.7	3.5259	3.3931	3.3002	0.1384	0.1439	0.1475	−0.0379	0	0.0250
0.9	2.4018	2.3209	2.2427	0.2194	0.2254	0.2317	−0.0267	0	0.0277
	$C_Q = 6$								
	r_{ISCO}/M			$M\zeta_{\mathrm{ISCO}}$			δ_{ISCO}		
$\hat{a}$	$\hat{s} = -0.1$	$\hat{s} = 0$	$\hat{s} = 0.1$	$\hat{s} = -0.1$	$\hat{s} = 0$	$\hat{s} = 0.1$	$\hat{s} = -0.1$	$\hat{s} = 0$	$\hat{s} = 0.1$
0	6.1893	6	5.8627	0.0656	0.0680	0.0698	−0.0356	0	0.0256
0.1	5.8565	5.6693	5.5346	0.0708	0.0735	0.0755	−0.0370	0	0.0264
0.3	5.1604	4.9786	4.8501	0.0841	0.0877	0.0901	−0.0402	0	0.0281
0.5	4.4065	4.2330	4.1122	0.1038	0.1086	0.1119	−0.0438	0	0.0301
0.7	3.5438	3.3931	3.2815	0.1375	0.1439	0.1486	−0.0446	0	0.0330
0.9	2.4094	2.3209	2.2504	0.2184	0.2254	0.2306	−0.0313	0	0.0230

ℓ_r, ℓ_ϕ and ℓ play the role of eccentric anomalies. Note that, for geodesics, the above quantities reduce to $a_r = r_0$, $e_t = e_r = e_\phi = 0$, $P = 2\pi\Gamma_\pm$, $\Phi = 2\pi\Omega_\pm$ and $\ell_t = \ell_r = \ell_\phi = \ell$. For nonvanishing spin parameter $\hat{s}$, the semimajor axis and the eccentricities for the orbit of the extended body turn out to be

$$a_r = r_0 - \hat{s}R_{\hat{s}} - \hat{s}^2 C_{1\hat{s}} \tag{4.46}$$

and

$$\begin{aligned} e_t &= -\frac{\Omega_{(\mathrm{ep})}}{\Gamma_\pm}\hat{s}\left[T_{\hat{s}} + \hat{s}\left(D_{1\hat{s}} + \frac{\Omega_{(\mathrm{ep})}}{\Gamma_\pm}T_{\hat{s}}^2\right)\right],\\ e_r &= -\frac{\hat{s}}{r_0}\left[R_{\hat{s}} + \hat{s}\left(C_{1\hat{s}} + \frac{R_{\hat{s}}^2}{r_0}\right)\right],\\ e_\phi &= \frac{\Omega_{(\mathrm{ep})}}{\Omega_\pm}\hat{s}\left[\bar{\zeta}_\pm T_{\hat{s}} + \hat{s}\left(E_{1\hat{s}} + \frac{\Omega_{(\mathrm{ep})}}{\Omega_\pm}\bar{\zeta}_\pm^2 T_{\hat{s}}^2\right)\right], \end{aligned} \tag{4.47}$$

respectively. As for the periods of t and ϕ motions, they may be written as

$$\begin{aligned} P &= 2\pi\frac{\Gamma_\pm}{\Omega_{(\mathrm{ep})}}\left[1 \mp 2\gamma_\pm\nu_\pm\frac{\zeta_K}{\Omega_{(\mathrm{ep})}}\mathcal{V}_{\hat{s}}^{(r)}\hat{s} + \frac{D_{4\hat{s}}}{\Gamma_\pm}\hat{s}^2\right],\\ \Phi &= 2\pi\frac{\Omega_\pm}{\Omega_{(\mathrm{ep})}}\left[1 \mp 2\gamma_\pm\nu_\pm\frac{\zeta_K}{\Omega_{(\mathrm{ep})}}\frac{\bar{\zeta}_\pm}{\zeta_\pm}\mathcal{V}_{\hat{s}}^{(r)}\hat{s} + \frac{E_{4\hat{s}}}{\Omega_\pm}\hat{s}^2\right]. \end{aligned} \tag{4.48}$$

There remains to display the three eccentric anomaly parameters, ℓ_t, ℓ_r and ℓ_ϕ, as functions of ℓ:

$$\begin{aligned} \ell_t &= \ell + \frac{\hat{s}^2}{\Gamma_\pm}[D_{2\hat{s}}\Omega_{(\mathrm{ep})}\sin 2\ell + D_{3\hat{s}}\ell\cos\ell],\\ \ell_r &= \ell + \frac{\hat{s}}{R_{\hat{s}}\Omega_{(\mathrm{ep})}}[2C_{2\hat{s}}\Omega_{(\mathrm{ep})}\sin\ell - C_{3\hat{s}}\ell],\\ \ell_\phi &= \ell + \frac{\hat{s}^2}{\Omega_\pm}\left[\Omega_{(\mathrm{ep})}\left(E_{2\hat{s}} - \frac{1}{4}\Omega_{(\mathrm{ep})}^2\bar{\zeta}_\pm^2 T_{\hat{s}}^3\right)\sin 2\ell\right.\\ &\quad\left. + E_{3\hat{s}}\ell\cos\ell\right]. \end{aligned} \tag{4.49}$$

The various coefficients entering the above formulas are listed in Appendix E. Notice that the parametrization (4.44) of the orbit is different from the quasi-Keplerian one, used in Refs. [60,61], due to the presence of different parameters representing the t, r and ϕ motions instead of a single eccentric anomaly. The two forms agree to first order in spin only, as shown in Ref. [42].

At this stage, we can derive explicit expressions for the conserved energy and angular momentum (2.15),

$$\begin{aligned} \tilde{E} &\equiv \frac{E}{m_0} = \tilde{E}_\pm + \hat{s}\tilde{E}_{\hat{s}} + \hat{s}^2\tilde{E}_{\hat{s}\hat{s}},\\ \tilde{J} &\equiv \frac{J}{m_0} = \tilde{L}_\pm + \hat{s}\tilde{J}_{\hat{s}} + \hat{s}^2\tilde{J}_{\hat{s}\hat{s}}, \end{aligned} \tag{4.50}$$

with

$$\tilde{E}_{\hat{s}} = M\Omega_\pm\left(\frac{M}{r_0} \mp a\zeta_K\right), \qquad \tilde{E}_{\hat{s}} - \zeta_\pm\tilde{J}_{\hat{s}} = \mp\frac{M\zeta_K}{\Gamma_\pm}, \tag{4.51}$$

and

$$\begin{aligned} \tilde{J}_{\hat{s}\hat{s}} &= -\sqrt{g_{\phi\phi}}\gamma_\pm^2\mathcal{V}_{\hat{s}}^{(r)}\left[(1-C_Q)M\Omega_{(\mathrm{ep})} + \frac{2}{\gamma_\pm\nu_\pm}\mathcal{V}_{\hat{s}}^{(r)}\right]\\ &\quad + \frac{m_2}{m_0}\tilde{L}_\pm,\\ \tilde{E}_{\hat{s}\hat{s}} - \zeta_\pm\tilde{J}_{\hat{s}\hat{s}} &= 2N\gamma_\pm(\mathcal{V}_{\hat{s}}^{(r)})^2 + \frac{m_2}{m_0}\frac{1}{\Gamma_\pm}, \end{aligned} \tag{4.52}$$

where $\tilde{E}_\pm$ and $\tilde{L}_\pm$ are the energy and the azimuthal angular momentum per unit (bare) mass for a circular geodesic, as given by Eqs. (3.18), the parameter m_2 representing the spin correction (4.8) to the mass of the body [see also Eq. (4.13)].

In the weak-field limit the previous expressions become

$$\begin{aligned} -2\hat{E} &\equiv -2(\tilde{E}-1) = u_0\left(1 - \frac{3}{4}u_0 - \frac{27}{8}u_0^2 - \frac{675}{64}u_0^3\right) + 2u_0^{5/2}\left(1 + \frac{9}{2}u_0 + \frac{135}{8}u_0^2\right)\hat{a}\\ &\quad - 2u_0^{5/2}\left(1 + \frac{3}{2}u_0 + \frac{27}{8}u_0^2\right)\hat{s} - u_0^3\left(1 + \frac{15}{2}u_0\right)\hat{a}^2 + 2u_0^3\left(1 + \frac{5}{2}u_0\right)\hat{a}\hat{s}\\ &\quad + u_0^3\left[C_Q + 6\left(1 + \frac{5}{12}C_Q\right)u_0\right]\hat{s}^2 + \mathcal{O}(u_0^5),\\ \hat{J} &\equiv \frac{\tilde{J}}{M} = u_0^{-1/2}\left(1 + \frac{3}{2}u_0 + \frac{27}{8}u_0^2 + \frac{135}{16}u_0^3 + \frac{2835}{128}u_0^4\right) - 3u_0\left(1 + \frac{5}{2}u_0 + \frac{63}{8}u_0^2 + \frac{405}{16}u_0^3\right)\hat{a}\\ &\quad + \left(1 - \frac{1}{2}u_0 + \frac{3}{8}u_0^2 + \frac{27}{16}u_0^3 + \frac{675}{128}u_0^4\right)\hat{s} + u_0^{3/2}\left(1 + 5u_0 + \frac{189}{8}u_0^2 + \frac{405}{4}u_0^3\right)\hat{a}^2 - 3u_0^{7/2}\hat{a}\hat{s}\\ &\quad - 21u_0^{5/2}\left[1 + \frac{1}{42}C_Q + \frac{1}{14}\left(71 + \frac{3}{2}C_Q\right)u_0 + \frac{15}{56}\left(111 + \frac{3}{2}C_Q\right)u_0^2\right]\hat{s}^2 + \mathcal{O}(u_0^5), \end{aligned} \tag{4.53}$$

for the corotating case. We list below the orbital elements (4.46)–(4.48) expressed in terms of the gauge invariant quantities $\widehat{E}$ and $\widehat{J}$, related by

$$\widehat{J} = (-2\widehat{E})^{-1/2}\left\{1+\frac{9}{8}(-2\widehat{E})+\frac{243}{128}(-2\widehat{E})^2+\frac{5373}{1024}(-2\widehat{E})^3-2(-2\widehat{E})^{3/2}\left[1+\frac{9}{4}(-2\widehat{E})+\frac{81}{8}(-2\widehat{E})^2\right]\widehat{a}\right.$$
$$+(-2\widehat{E})^{1/2}\left[1-\frac{3}{2}(-2\widehat{E})-\frac{9}{4}(-2\widehat{E})^2-\frac{81}{8}(-2\widehat{E})^3\right]\widehat{s}+\frac{1}{2}(-2\widehat{E})^2\left[1+\frac{69}{8}(-2\widehat{E})\right]\widehat{a}^2$$
$$\left.+(-2\widehat{E})^2\left[1+\frac{45}{8}(-2\widehat{E})\right]\widehat{a}\,\widehat{s}+\frac{1}{2}(-2\widehat{E})^2\left[C_Q-3\left(15-\frac{7}{8}C_Q\right)(-2\widehat{E})\right]\widehat{s}^2+\mathcal{O}[(-2\widehat{E})^4]\right\}, \tag{4.54}$$

so that we may write them with the help of a single parameter, e.g., the energy parameter $(-2\widehat{E})\equiv X$. We find

$$\frac{a_r}{M} = X^{-1}\left(1-\frac{3}{4}X-\frac{63}{16}X^2-\frac{1215}{64}X^3-\frac{27135}{256}X^4\right)+X^{1/2}\left(1+\frac{51}{8}X+\frac{5751}{128}X^2\right)(2\widehat{a}+\widehat{s})$$
$$-X\left(1+\frac{63}{4}X+183X^2\right)\widehat{a}^2-X\left(1+\frac{79}{4}X+213X^2\right)\widehat{a}\,\widehat{s}$$
$$-\frac{1}{2}X\left[C_Q-\left(36+\frac{5}{4}C_Q\right)X-3\left(86-\frac{1}{2}C_Q\right)X^2\right]\widehat{s}^2+\mathcal{O}(X^{7/2}),$$
$$e_t = -6X^{5/2}\left\{\left(1+\frac{63}{8}X\right)\widehat{s}-X^{1/2}\left(1+\frac{97}{4}X\right)\widehat{a}\,\widehat{s}-\frac{1}{2}X^{1/2}\left[C_Q+\left(1+\frac{25}{4}C_Q\right)X\right]\widehat{s}^2+\mathcal{O}(X^2)\right\},$$
$$e_r = 3X^{3/2}\left\{\left(1+\frac{41}{8}X\right)\widehat{s}-X^{1/2}\left(1+\frac{33}{2}X\right)\widehat{a}\,\widehat{s}-\frac{1}{2}X^{1/2}\left[C_Q-\left(10-\frac{7}{2}C_Q\right)X\right]\widehat{s}^2+\mathcal{O}(X^2)\right\},$$
$$e_\phi = 6X^{3/2}\left\{\left(1+\frac{41}{8}X\right)\widehat{s}-X^{1/2}\left(1+\frac{31}{2}X\right)\widehat{a}\,\widehat{s}-\frac{1}{2}X^{1/2}\left[C_Q-\left(4-\frac{7}{2}C_Q\right)X\right]\widehat{s}^2+\mathcal{O}(X^2)\right\}. \tag{4.55}$$

At last, the periods of t and ϕ motions are given by

$$\frac{1}{M}\frac{P}{2\pi} = X^{-3/2}\left\{1+\frac{15}{8}X+\frac{855}{128}X^2+\frac{41175}{1024}X^3-12X^{5/2}\left(1+\frac{51}{4}X\right)\widehat{a}-3X^{3/2}(1+6X+39X^2)\widehat{s}\right.$$
$$\left.+11X^3\widehat{a}^2+3X^2\left(1+\frac{99}{8}X\right)\widehat{a}\,\widehat{s}+\frac{3}{2}X^2\left[C_Q+\left(1+\frac{59}{8}C_Q\right)X\right]\widehat{s}^2+\mathcal{O}(X^4)\right\},$$
$$\frac{\Phi}{2\pi} = 1+3X+\frac{63}{4}X^2+\frac{405}{4}X^3-4X^{3/2}\left(1+\frac{93}{8}X\right)\widehat{a}-6X^{3/2}\left(1+\frac{57}{8}X\right)\widehat{s}$$
$$+\frac{3}{2}X^2\left(1+\frac{73}{2}X\right)\widehat{a}^2+6X^2\left(1+\frac{39}{2}X\right)\widehat{a}\,\widehat{s}+3X^2\left[C_Q+\frac{1}{2}(13+11C_Q)X\right]\widehat{s}^2+\mathcal{O}(X^{7/2}). \tag{4.56}$$

V. CONCLUDING REMARKS

We have investigated finite-size effects on the motion of extended test bodies, in the equatorial plane of a Kerr spacetime, within the framework of the Mathisson-Papapetrou-Dixon model up to the quadrupolar order. In general, the quadrupole tensor shares the same symmetries as the Riemann tensor and is completely specified by two symmetric, trace-free spatial tensors, i.e., the mass quadrupole (electric) and the current quadrupole (magnetic) tensors, whose role has been investigated in previous works [42,43]. Here we have considered the rotational deformation induced by a quadrupole tensor of the electric-type only, taken to be proportional to the trace-free part of the square of the spin tensor, with a constant proportionality parameter which may be regarded as the polarizability of the object. This allows us to treat on an equal footing the cases of black holes and neutron stars, thus generalizing previous works.

The general features of equatorial motion have been discussed through the analysis of the associated radial effective potentials. We have obtained their generalization from the well-known case of a co- or counter-rotating test monopole particle in a Kerr spacetime to that of an extended test body with spin-induced quadrupole moment. We have also evaluated the correction to the ISCO due to spin and the corresponding frequency, which is an important observable in gravitational-wave astronomy. The presence of spin corrections introduce an uncertainty on the values of the corresponding quantities for structureless particles. On the other hand, those features can be used to determine whether the small object is endowed with a spin, by performing an adequate parameter estimation in the context of gravitational-wave detection.

The dynamics of the system have been studied not only qualitatively, but also quantitatively. In fact, neglecting terms in the MPD equations that are of third order in spin or higher allowed us to solve the problem in a fully analytic way. Initial conditions have been chosen so that the tangent vector to the orbit of the extended body be initially tangent to the 4-velocity of a timelike spatially circular geodesic, taken as the reference trajectory. We have obtained the "perturbative" solution to second order in spin in the following two cases: (i) when the trajectory of the extended body remains circular with spin-dependent frequency and (ii) when it deviates from circular motion because of the combined effects of both the spin-curvature and quadrupole-curvature couplings (i.e., when the orbit is "quasicircular"). The tangent vector to the orbit and the unit timelike vector aligned with the 4-momentum are, in general, distinct. However, there exists a special value of the polarizability constant, which corresponds to the black hole case, such that they are aligned not only initially, but all along the (circular) trajectory of the extended body. This is no longer true for neutron stars, an interesting fact which seems to have never been pointed out before. For quasicircular orbits, we have explicitly written down the solution in a Keplerian-like form, by introducing the temporal, radial and azimuthal eccentricities of the orbit, as well as the associated periods and frequencies. We have also computed the spin-induced shift of the conserved energy and angular momentum, in a gauge-invariant way. All orbital elements have been expanded in the weak-field and slow-motion limit, in a more suitable form to be compared with the existing post-Newtonian literature.

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APPENDIX A: $1+3$ DECOMPOSITION OF THE QUADRUPOLE TENSOR

Let us consider an orthonormal frame adapted to an observer family characterized by the 4-velocity u (with normalization $u\cdot u=-1$), say $e_0=u$ and $\{e_a\}$, $a=1,2,3$, a triad of three unit spatial vectors orthogonal to u. We shall introduce the compact notation $X_{\alpha\beta\ldots}u^\alpha = X_{u\beta\ldots}$ for tensor contraction. In addition, we shall denote by ${}^*X_{\rho\sigma\ldots}=\frac{1}{2}\eta^{\alpha\beta}{}_{\rho\sigma}X_{\alpha\beta\ldots}$ and $X^*_{\ldots\mu\nu}=\frac{1}{2}X_{\ldots\gamma\delta}\eta^{\gamma\delta}{}_{\mu\nu}$ the left and right dual of a tensor, respectively. The standard $1+3$ decomposition of the Riemann tensor in terms of its electric (spatial and symmetric) part $E(u)$, its magnetic (spatial and trace-free) part $H(u)$, and its mixed (spatial and symmetric) part $F(u)$, defined by

$$E(u)_{\alpha\beta}=R_{\alpha u\beta u},\quad H(u)_{\alpha\beta}=-R^*_{\alpha u\beta u},\quad F(u)_{\alpha\beta}={}^*R^*_{\alpha u\beta u}, \tag{A1}$$

respectively, leads to the identification of the 20 original independent components: 6 in $E(u)$, 8 in $H(u)$ and 6 in $F(u)$.

Similarly, since the algebraic symmetries of the quadrupole $J_{\alpha\beta\gamma\delta}$ are the same as for $R_{\alpha\beta\gamma\delta}$, one can decompose the former quantity in terms of the associated tensors

$$\mathcal{Q}(u)_{\alpha\beta}=J_{\alpha u\beta u},\quad \mathcal{W}(u)_{\alpha\beta}=-J^*_{\alpha u\beta u},\quad \mathcal{M}(u)_{\alpha\beta}={}^*J^*_{\alpha u\beta u}. \tag{A2}$$

In so doing, we identify its electric (spatial and symmetric) part $\mathcal{Q}(u)$, with 6 independent components, its magnetic (spatial and trace-free) part $\mathcal{W}(u)$, with 8 independent components, and its mixed (spatial and symmetric) part $\mathcal{M}(u)$, with 6 independent components. However, J enters the MPD dynamics only in certain combinations, through specific contractions with the Riemann tensor or its derivative. Hence, the number of effective components needed is reduced by half, as shown in detail in Refs. [41,42]. The proof requires the replacement of the mixed part $\mathcal{M}(u)$ by a new tensor $\mathcal{X}(u)$ (with the same symmetries), according to

$$\mathcal{M}(u)=\mathcal{Q}(u)+\mathcal{X}(u), \tag{A3}$$

as well as the decomposition of both $\mathcal{X}(u)$ and $\mathcal{W}(u)$ in terms of their STF and pure-trace parts,

$$\begin{aligned}\mathcal{X}(u)&=\mathcal{X}(u)^{\rm STF}+\frac{1}{3}[{\rm Tr}\mathcal{X}(u)]P(u),\\ \mathcal{W}(u)&=\mathcal{W}(u)^{\rm STF}+\frac{1}{3}[{\rm Tr}\mathcal{W}(u)]P(u),\end{aligned} \tag{A4}$$

where $[P(u)]^\alpha{}_\beta=\delta^\alpha{}_\beta+u^\alpha u_\beta$ denotes the projector to the hyperplane orthogonal to u. Inserting the resulting expression for J into the equations of motion then cancels the contribution of $\mathcal{Q}(u)$, which yields the following "effective" representation of the quadrupole tensor (valid only in the context of the MPD model):

$$\begin{aligned}J^{\alpha\beta}{}_{\gamma\delta}&=\eta(u)^{\alpha\beta\mu}\eta(u)_{\gamma\delta}{}^{\nu}[\mathcal{X}(u)]^{\rm STF}_{\mu\nu}+2u^{[\alpha}[\mathcal{W}(u)]^{{\rm STF}\beta]}{}_\sigma\eta(u)^\sigma{}_{\gamma\delta}\\ &\quad+2u_{[\gamma}[\mathcal{W}(u)]^{\rm STF}{}_{\delta]\sigma}\eta(u)^\sigma{}_{\alpha\beta},\end{aligned} \tag{A5}$$

with $\eta(u)_{\alpha\beta\gamma}=u^\mu\eta_{\mu\alpha\beta\gamma}$ defining the space 3-volume form (see Sec. II). Summarizing, in basis components, we can write

$$\begin{aligned}J^{0a}{}_{0b}&=[\mathcal{X}(u)^{\rm STF}]^a{}_b,\\ J^{0a}{}_{bc}&=[\mathcal{W}(u)^{\rm STF}]^a{}_d\eta(u)^d{}_{bc}=[\mathcal{W}(u)^{\rm STF}]^{*(u)a}{}_{bc},\\ J^{ab}{}_{cd}&=\eta(u)^{abr}\eta(u)_{cd}{}^s[\mathcal{X}(u)^{\rm STF}]_{rs}=[{}^{*(u)}[\mathcal{X}(u)^{\rm STF}]^{*(u)}]^{ab}{}_{cd}.\end{aligned} \tag{A6}$$

For convenience, we actually use the following notation:

$$\mathcal{X}(u)^{\rm STF} = \widetilde{\mathcal{X}}(u), \qquad \mathcal{W}(u)^{\rm STF} = \widetilde{\mathcal{W}}(u). \tag{A7}$$

APPENDIX B: 1 + 3 DECOMPOSITION OF THE MPD EQUATIONS

It is useful to perform a 1 + 3 splitting with respect to U of the MPD equations (2.1)–(2.2). A key observation is that the force term on the right-hand side of the first equation (2.1) is not spatial for the comoving observer with 4-velocity U, since $F_{\rm (spin)}\cdot U = 0$ whereas $F_{\rm (quad)}\cdot U \neq 0$. Recalling that the operator $P(U)^\alpha{}_\beta = \delta^\alpha{}_\beta + U^\alpha U_\beta$ represents a projector perpendicularly to U, we see that

$$\begin{aligned}\frac{DP^\mu}{d\tau} &= F^\mu_{\rm (spin)} + [P(U)F_{\rm (quad)}]^\mu + \frac{1}{6}U^\mu J^{\alpha\beta\gamma\delta}\frac{D}{d\tau}R_{\alpha\beta\gamma\delta}\\ &\equiv F(U)^\mu_{\rm (tot)} + U^\mu\frac{dm_J}{d\tau} - \frac{1}{6}U^\mu R_{\alpha\beta\gamma\delta}\frac{D}{d\tau}J^{\alpha\beta\gamma\delta},\end{aligned} \tag{B1}$$

where the force

$$\begin{aligned}F(U)^\mu_{\rm (tot)} &\equiv F^\mu_{\rm (spin)} + [P(U)F_{\rm (quad)}]^\mu\\ &= [P(U)(F_{\rm (spin)} + F_{\rm (quad)})]^\mu\end{aligned} \tag{B2}$$

is orthogonal to U and the mass correction m_J has been defined in Eq. (2.13). In a second stage, we get from Eq. (B1),

$$\frac{DP^\mu}{d\tau} - U^\mu\frac{dm_J}{d\tau} = F(U)^\mu_{\rm (tot)} - \frac{1}{6}U^\mu R_{\alpha\beta\gamma\delta}\frac{D}{d\tau}J^{\alpha\beta\gamma\delta}, \tag{B3}$$

or, equivalently,

$$\begin{aligned}U_\mu\frac{DP^\mu}{d\tau} + \frac{dm_J}{d\tau} &= \frac{1}{6}R_{\alpha\beta\gamma\delta}\frac{D}{d\tau}J^{\alpha\beta\gamma\delta},\\ P(U)\left[\frac{DP^\mu}{d\tau}\right] &= F(U)^\mu_{\rm (tot)},\end{aligned} \tag{B4}$$

which follows from projecting Eq. (B3) along U and perpendicularly to U.

APPENDIX C: ZAMO RELEVANT QUANTITIES

We list below the nonvanishing components of the electric and magnetic parts of the Riemann tensor, as well as the relevant kinematical quantities as measured by ZAMOs and evaluated in the equatorial plane.

The radial components of the acceleration and expansion vectors are given by

$$\begin{aligned}a(n)^{\hat r} &= \frac{M}{r^2\sqrt{\Delta}}\frac{(r^2+a^2)^2 - 4a^2Mr}{r^3 + a^2r + 2a^2M},\\ \theta_{\hat\phi}(n)^{\hat r} &= -\frac{aM(3r^2+a^2)}{r^2(r^3+a^2r+2a^2M)},\end{aligned} \tag{C1}$$

respectively. The expressions for the radial components of the curvature vector are

$$\begin{aligned}&\kappa(r,n)^{\hat r} = \frac{Mr - a^2}{r^2\sqrt{\Delta}}, \qquad \kappa(\theta,n)^{\hat r} = -\frac{\sqrt{\Delta}}{r^2},\\ &k_{\rm (Lie)} = -\frac{(r^3 - a^2M)\sqrt{\Delta}}{r^2(r^3+a^2r+2a^2M)}.\end{aligned} \tag{C2}$$

Finally, the nontrivial components of the electric and magnetic parts of the Riemann tensor with respect to ZAMOs read

$$\begin{aligned}E_{\hat r\hat r} &= -\frac{M(2r^4 + 5r^2a^2 - 2a^2Mr + 3a^4)}{r^4(r^3+a^2r+2a^2M)},\\ E_{\hat\theta\hat\theta} &= -E_{\hat\phi\hat\phi} - E_{\hat r\hat r}, \qquad E_{\hat\phi\hat\phi} = \frac{M}{r^3},\\ H_{\hat r\hat\theta} &= -\frac{3Ma(r^2+a^2)\sqrt{\Delta}}{r^4(r^3+a^2r+2a^2M)}.\end{aligned} \tag{C3}$$

In the limit of the vanishing rotation parameter ($a\to 0$), the previous quantities reduce to

$$\begin{aligned}&a(n)^{\hat r} = \frac{M}{Nr^2}, \qquad \theta_{\hat\phi}(n)^{\hat r} = 0, \qquad k_{\rm (Lie)} = -\frac{N}{r},\\ &E_{\hat r\hat r} = -\frac{2M}{r^3}, \qquad H_{\hat r\hat\theta} = 0,\end{aligned} \tag{C4}$$

with $N = \sqrt{1 - 2M/r}$.

APPENDIX D: FRAME COMPONENTS OF BOTH SPIN AND QUADRUPOLE TERMS

We list below the explicit expressions for the components of both spin and quadrupole terms with respect to the frame adapted to u.

The spin force is given by Eq. (3.25) with

$$\begin{aligned}F^1_{\rm (spin)} &= s\gamma\gamma_u\{\nu_u\cos 2\alpha_u E_{\hat r\hat r} + [\nu\cos(\alpha_u-\alpha) + \nu_u\cos^2\alpha_u]E_{\hat\theta\hat\theta} + \sin\alpha_u[1 + \nu\nu_u\cos(\alpha_u-\alpha)]H_{\hat r\hat\theta}\},\\ F^2_{\rm (spin)} &= -{\rm sgn}(\nu_u)s\gamma\gamma_u^2\{\nu_u[\sin 2\alpha_u - \nu\nu_u\sin(\alpha_u+\alpha)]E_{\hat r\hat r} + [\nu\sin(\alpha_u-\alpha) + \nu_u\cos\alpha_u(\sin\alpha_u - \nu\nu_u\sin\alpha)]E_{\hat\theta\hat\theta}\\ &\quad - [\cos\alpha_u(1 + \nu\nu_u\cos(\alpha_u-\alpha)) - 2\nu\nu_u\cos\alpha)]H_{\hat r\hat\theta}\},\end{aligned} \tag{D1}$$

the remaining component $F^0_{(\mathrm{spin})}$ following from the condition $F_{(\mathrm{spin})}\cdot U=0$, i.e.,

$$\gamma_u(1-\nu\nu_u\cos(\alpha_u-\alpha))F^0_{(\mathrm{spin})}=F^1_{(\mathrm{spin})}\nu\sin(\alpha_u-\alpha)+\mathrm{sgn}(\nu_u)F^2_{(\mathrm{spin})}\gamma_u(-\nu_u+\nu\cos(\alpha_u-\alpha)). \tag{D2}$$

On the other hand, the spin quantity $D_{(\mathrm{spin})}$ takes the form of Eq. (3.26), with

$$\mathcal{E}_{(\mathrm{spin})}(u)=m\gamma[\nu\sin(\alpha_u-\alpha)\omega^1+\mathrm{sgn}(\nu_u)\gamma_u(\nu\cos(\alpha_u-\alpha)-\nu_u)\omega^2]. \tag{D3}$$

Concerning the quadrupole contributions, one gets for the components of the force (3.25):

$$\begin{aligned}
F^1_{(\mathrm{quad})}&=-\frac{1}{4}\frac{s^2}{m}C_Q\left\{\gamma_u^2\nu_u[b_1\nu_u\sin3\alpha_u+2(b_4\cos2\alpha_u+b_5)]-\left[(b_1-2b_2)\gamma_u^2-b_1+\frac{2}{3}(b_2+b_3)\right]\sin\alpha_u\right\},\\
F^2_{(\mathrm{quad})}&=-\mathrm{sgn}(\nu_u)\frac{1}{4}\frac{s^2}{m}C_Q\gamma_u\left\{b_1\gamma_u^2\nu_u^2\cos3\alpha_u+2[(a_1-b_4)\gamma_u^2-a_1]\nu_u\sin2\alpha_u\right.\\
&\quad\left.+\left[(b_1+2c_3)\gamma_u^2-b_1-\frac{2}{3}(c_1+c_2)\right]\cos\alpha_u\right\},
\end{aligned} \tag{D4}$$

with

$$\begin{aligned}
a_1&=(E_{\hat\theta\hat\theta}+2E_{\hat r\hat r})\theta_{\hat\phi}(n)^{\hat r}+H_{\hat r\hat\theta}a(n)^{\hat r},\\
a_2&=(2E_{\hat\theta\hat\theta}+E_{\hat r\hat r})a(n)^{\hat r}-H_{\hat r\hat\theta}\theta_{\hat\phi}(n)^{\hat r},\\
b_1&=-2(E_{\hat\theta\hat\theta}+2E_{\hat r\hat r})k_{(\mathrm{Lie})}+(2E_{\hat\theta\hat\theta}+E_{\hat r\hat r})a(n)^{\hat r}-4H_{\hat r\hat\theta}\theta_{\hat\phi}(n)^{\hat r}+\frac{1}{2}\partial_{\hat r}E_{\hat\theta\hat\theta},\\
b_2&=-(E_{\hat\theta\hat\theta}+2E_{\hat r\hat r})k_{(\mathrm{Lie})}+2H_{\hat r\hat\theta}\theta_{\hat\phi}(n)^{\hat r}-\frac{3}{2}\partial_{\hat r}E_{\hat\theta\hat\theta},\\
b_3&=-2(E_{\hat\theta\hat\theta}+2E_{\hat r\hat r})k_{(\mathrm{Lie})}-2H_{\hat r\hat\theta}\theta_{\hat\phi}(n)^{\hat r},\\
b_4&=H_{\hat r\hat\theta}k_{(\mathrm{Lie})}-3E_{\hat\theta\hat\theta}\theta_{\hat\phi}(n)^{\hat r}+\partial_{\hat r}H_{\hat r\hat\theta},\\
b_5&=H_{\hat r\hat\theta}k_{(\mathrm{Lie})}-(E_{\hat\theta\hat\theta}+2E_{\hat r\hat r})\theta_{\hat\phi}(n)^{\hat r}-\partial_{\hat r}H_{\hat r\hat\theta},
\end{aligned} \tag{D5}$$

as well as $c_1=c_3+2a_2$, $c_2=c_3-b_2$ and $c_3=[7b_2+4a_2+6(b_1-2b_3)]/5$, whereas $F^0_{(\mathrm{quad})}$ may be obtained by requiring that the coordinate component $F_{(\mathrm{quad})t}=0$ vanishes, which implies

$$0=\gamma_u(F^0_{(\mathrm{quad})}+\mathrm{sgn}(\nu_u)\nu_uF^2_{(\mathrm{quad})})+\frac{\sqrt{g_{\phi\phi}}N^\phi}{N}[F^1_{(\mathrm{quad})}\cos\alpha_u-\gamma_u\sin\alpha_u(\nu_uF^0_{(\mathrm{quad})}+\mathrm{sgn}(\nu_u)F^2_{(\mathrm{quad})})]. \tag{D6}$$

Finally, the torque tensor $D^{\mu\nu}_{(\mathrm{quad})}$, whose structure is displayed in Eqs. (3.26)–(3.27), is such that

$$\begin{aligned}
\mathcal{E}_{(\mathrm{quad})\,1}&=\frac{s^2}{m}C_Q\gamma_u\cos\alpha_u[\nu_u\sin\alpha_u(2E_{\hat r\hat r}+E_{\hat\theta\hat\theta})-H_{\hat r\hat\theta}],\\
\mathcal{E}_{(\mathrm{quad})\,2}&=\mathrm{sgn}(\nu_u)\frac{s^2}{m}C_Q\gamma_u^2[\nu_u\cos2\alpha_uE_{\hat r\hat r}+\nu_u(1+\cos^2\alpha_u)E_{\hat\theta\hat\theta}+\sin\alpha_u(1+\nu_u^2)H_{\hat r\hat\theta}].
\end{aligned} \tag{D7}$$

1. The Schwarzschild limit

We list below the corresponding expressions of both spin and quadrupole terms in the limit of vanishing Kerr spin parameter.

The spin force (3.25) becomes

$$F_{(\mathrm{spin})} = \frac{M}{r^3}\gamma\gamma_u s\Big\{3\gamma_u\nu_u\sin\alpha_u(\nu\cos\alpha-\nu_u\cos\alpha_u)u + \frac{1}{2}[\nu_u(1-3\cos 2\alpha_u)+2\nu\cos(\alpha_u-\alpha)]e_1 - \frac{1}{2}\mathrm{sgn}(\nu_u)\gamma_u[(2+\nu_u^2)\nu\sin(\alpha_u-\alpha) + 3\nu_u(\nu\nu_u\sin(\alpha_u+\alpha)-\sin 2\alpha_u)]e_2\Big\}, \tag{D8}$$

whereas the spin quantity $D_{(\mathrm{spin})}$ is still given by Eq. (3.26) with components (D3).

The quadrupole force (3.25) reduces to

$$F_{(\mathrm{quad})} = F^1_{(\mathrm{quad})}e_1 + F^2_{(\mathrm{quad})}[-\mathrm{sgn}(\nu_u)\nu_u u + e_2], \tag{D9}$$

where $\gamma_u[-\mathrm{sgn}(\nu_u)\nu_u u + e_2]$ represents a unitary and spacelike vector orthogonal to n and

$$F^1_{(\mathrm{quad})} = -\frac{3MN}{2r^4}\frac{s^2}{m}C_Q\gamma_u^2\sin\alpha_u\left[1-\frac{1}{2}\nu_u^2(1+5\cos 2\alpha_u)\right],$$
$$F^2_{(\mathrm{quad})} = -\mathrm{sgn}(\nu_u)\frac{3MN}{2r^4}\frac{s^2}{m}C_Q\gamma_u^2 \times\cos\alpha_u\left[1+\frac{1}{2}\nu_u^2(3-5\cos 2\alpha_u)\right]. \tag{D10}$$

Finally, for the torque term as shown in Eq. (3.26), we have

$$\mathcal{E}_{(\mathrm{quad})} = -\frac{3M}{2r^3}\frac{s^2}{m}C_Q\gamma_u\nu_u\sin 2\alpha_u[\omega^1 - \mathrm{sgn}(\nu_u)\gamma_u\tan\alpha_u\omega^2]. \tag{D11}$$

APPENDIX E: QUASICIRCULAR ORBITS: COEFFICIENTS

We list below the various coefficients entering the quasicircular orbit solution (4.44):

$$R_{\hat s} = -\gamma_\pm\frac{\sqrt{\Delta}}{r_0\Omega_{(\mathrm{ep})}}\mathcal{V}^{(r)}_{\hat s},\qquad T_{\hat s} = \pm 2\frac{\gamma_\pm^2\nu_\pm}{N}\frac{\zeta_K}{\Omega^2_{(\mathrm{ep})}}\mathcal{V}^{(r)}_{\hat s}, \tag{E1}$$

with

$$\mathcal{V}^{(r)}_{\hat s} = -3\frac{M\sqrt{\Delta}}{r_0\Omega_{(\mathrm{ep})}}\frac{\gamma_\pm\zeta_\pm^2}{N^2}\left(\frac{a}{r_0}\mp r_0\zeta_K\right), \tag{E2}$$

and

$$\begin{aligned}
C_{1\hat s} &= \gamma_\pm\frac{\sqrt{\Delta}}{r_0\Omega^2_{(\mathrm{ep})}}\{\nu_\pm(B_{1\hat s}\Omega_{(\mathrm{ep})}-B_{3\hat s})+\gamma_\pm(\mathcal{V}^{(r)}_{\hat s})^2(\kappa(r,n)^{\hat r}+k_{(\mathrm{Lie})})\},\\
C_{2\hat s} &= \gamma_\pm\frac{\sqrt{\Delta}}{2r_0\Omega_{(\mathrm{ep})}}\left\{\nu_\pm B_{2\hat s}-\frac{\gamma_\pm}{2\Omega_{(\mathrm{ep})}}(\kappa(r,n)^{\hat r}+k_{(\mathrm{Lie})})(\mathcal{V}^{(r)}_{\hat s})^2\right\},\\
C_{3\hat s} &= -\gamma_\pm\nu_\pm\frac{\sqrt{\Delta}}{r_0\Omega_{(\mathrm{ep})}}B_{3\hat s},\\
D_{1\hat s} &= \mp 2\gamma_\pm^2\nu_\pm^2\frac{\zeta_K}{N\Omega^3_{(\mathrm{ep})}}(B_{1\hat s}\Omega_{(\mathrm{ep})}-2B_{3\hat s})+\frac{\mathcal{V}^{(r)}_{\hat s}}{MN\Omega^2_{(\mathrm{ep})}}\Big\{\gamma_\pm^2\nu_\pm M^2(\zeta_K^2-\Omega^2_{(\mathrm{ep})})\\
&\quad + M\Omega_{(\mathrm{ep})}\gamma_\pm^3\mathcal{V}^{(r)}_{\hat s}\left[1\mp\frac{2\nu_\pm\zeta_K}{\Omega^2_{(\mathrm{ep})}}((3\nu_\pm^2-4)k_{(\mathrm{Lie})}\pm 6\nu_\pm\zeta_K)\right]+2\gamma_\pm^4\nu_\pm(\mathcal{V}^{(r)}_{\hat s})^2\Big\},\\
D_{2\hat s} &= \mp\gamma_\pm^2\nu_\pm^2\frac{\zeta_K}{2N\Omega^2_{(\mathrm{ep})}}B_{2\hat s}-\frac{\gamma_\pm^3}{4N\Omega^2_{(\mathrm{ep})}}(\mathcal{V}^{(r)}_{\hat s})^2\left\{\frac{\gamma_\pm\nu_\pm^2}{4M\Omega_{(\mathrm{ep})}}\mathcal{V}^{(r)}_{\hat s}+1\mp\frac{\nu_\pm\zeta_K}{\Omega^2_{(\mathrm{ep})}}[(3\nu_\pm^2-4)k_{(\mathrm{Lie})}\pm 6\nu_\pm\zeta_K]\right\},\\
D_{3\hat s} &= \mp 2\gamma_\pm^2\nu_\pm^2\frac{\zeta_K}{N\Omega^2_{(\mathrm{ep})}}B_{3\hat s},\qquad D_{4\hat s} = -\Omega_{(\mathrm{ep})}(D_{1\hat s}+2D_{2\hat s})-D_{3\hat s},\\
E_{1\hat s} &= \bar\zeta_\pm D_{1\hat s}-\frac{\gamma_\pm}{\nu_\pm\sqrt{g_{\phi\phi}}\Omega_{(\mathrm{ep})}}(\mathcal{V}^{(r)}_{\hat s})^2,\qquad E_{2\hat s} = \bar\zeta_\pm D_{2\hat s}+\frac{\gamma_\pm}{4\nu_\pm\sqrt{g_{\phi\phi}}\Omega_{(\mathrm{ep})}}(\mathcal{V}^{(r)}_{\hat s})^2,\\
E_{3\hat s} &= \bar\zeta_\pm D_{3\hat s},\qquad E_{4\hat s} = \bar\zeta_\pm D_{4\hat s}+\frac{\gamma_\pm}{2\nu_\pm\sqrt{g_{\phi\phi}}}(\mathcal{V}^{(r)}_{\hat s})^2,
\end{aligned} \tag{E3}$$

where

$$
\begin{aligned}
B_{1\hat{s}} &= -2B_{2\hat{s}} - \frac{B_{3\hat{s}}}{\Omega_{(\mathrm{ep})}} - \frac{1}{\gamma_\pm \nu_\pm \Omega_{(\mathrm{ep})}}(\tilde{F}^1_{(\mathrm{quad})} \pm \zeta_K \tilde{\mathcal{E}}^2_{(\mathrm{quad})}) + \frac{\mathcal{V}^{(r)}_{\hat{s}}}{\nu_\pm}\left[\frac{\gamma^2_\pm}{\nu_\pm} \pm M\zeta_K + \frac{4\gamma^2_\pm}{3M\Omega^2_{(\mathrm{ep})}}(\nu_\pm k_{(\mathrm{Lie})} \mp 3\zeta_K)(\mathcal{V}^{(r)}_{\hat{s}})^2\right],\\
B_{2\hat{s}} &= (\mathcal{V}^{(r)}_{\hat{s}})^2\left\{\frac{\gamma^2_\pm}{3\nu_\pm M\Omega^2_{(\mathrm{ep})}}(\nu_\pm k_{(\mathrm{Lie})} \mp 3\zeta_K)\mathcal{V}^{(r)}_{\hat{s}} + \frac{2\zeta^2_K}{\gamma_\pm \nu_\pm \Omega^3_{(\mathrm{ep})}}(k_{(\mathrm{Lie})} \mp 2\gamma^2_\pm \nu_\pm \zeta_K)\right.\\
&\quad \left. - \frac{\gamma_\pm}{2\nu_\pm}\frac{k_{(\mathrm{Lie})}}{\Omega_{(\mathrm{ep})}} - \frac{\gamma^3_\pm}{6\nu_\pm \Omega^3_{(\mathrm{ep})}}[\nu^2_\pm \partial_{\hat{r}} E_{\hat{\theta}\hat{\theta}} + 2\nu_\pm \partial_{\hat{r}} H_{\hat{r}\hat{\theta}} - \partial_{\hat{r}} E_{\hat{r}\hat{r}}]\right\},\\
B_{3\hat{s}} &= -3\Omega_{(\mathrm{ep})}B_{2\hat{s}} + \frac{\mathcal{V}^{(r)}_{\hat{s}}}{\nu_\pm}\left\{\frac{2\gamma^2_\pm}{M\Omega_{(\mathrm{ep})}}(\nu_\pm k_{(\mathrm{Lie})} \mp 3\zeta_K)(\mathcal{V}^{(r)}_{\hat{s}})^2 - \frac{3}{2}\gamma_\pm k_{(\mathrm{Lie})}\mathcal{V}^{(r)}_{\hat{s}} + \frac{M\zeta^2_K}{\Omega_{(\mathrm{ep})}}(5\nu_\pm k_{(\mathrm{Lie})} \mp 2\zeta_K)\right.\\
&\quad \left. - M\Omega_{(\mathrm{ep})}(\nu_\pm k_{(\mathrm{Lie})} \mp \zeta_K) + \frac{M\gamma^2_\pm}{2\Omega_{(\mathrm{ep})}}[(\nu^2_\pm + 1)\partial_{\hat{r}} H_{\hat{r}\hat{\theta}} - \nu_\pm \partial_{\hat{r}}(E_{\hat{r}\hat{r}} - E_{\hat{\theta}\hat{\theta}})]\right\},
\end{aligned}
\tag{E4}
$$

and

$$
\begin{aligned}
\tilde{F}^1_{(\mathrm{quad})} &= \frac{1}{2}C_Q\left\{\gamma^2_\pm M^2[(1+\nu^2_\pm)\partial_{\hat{r}} E_{\hat{\theta}\hat{\theta}} + 2\nu_\pm \partial_{\hat{r}} H_{\hat{r}\hat{\theta}}] + 2\frac{M\Omega_{(\mathrm{ep})}\gamma_\pm \mathcal{V}^{(r)}_{\hat{s}}}{1+\nu^2_\pm}(\nu_\pm k_{(\mathrm{Lie})} \pm \zeta_K)(1+2\nu^2_\pm)\right.\\
&\quad \left. - M^2\left[2E_{\hat{\theta}\hat{\theta}} + E_{\hat{r}\hat{r}} + \frac{E_{\hat{r}\hat{r}} - E_{\hat{\theta}\hat{\theta}}}{1+\nu^2_\pm}\right](k_{(\mathrm{Lie})} \mp 2\gamma^2_\pm \nu_\pm \zeta_K)\nu^2_\pm\right\},\\
\tilde{\mathcal{E}}^2_{(\mathrm{quad})} &= C_Q M\Omega_{(\mathrm{ep})}\gamma_\pm \mathcal{V}^{(r)}_{\hat{s}}.
\end{aligned}
\tag{E5}
$$

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