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In the case involving particles, the necessary and sufficient condition for the Penrose process to extract energy from a rotating black hole is absorption of particles with negative energies and angular momenta. No torque at the black-hole horizon occurs. In this article we consider the case of arbitrary fields or matter described by an unspecified, general energy-momentum tensor $T_{\mu\nu}$ and show that the necessary and sufficient condition for extraction of a black hole's rotational energy is analogous to that in the mechanical Penrose process: absorption of negative energy and negative angular momentum. We also show that a necessary condition for the Penrose process to occur is for the Noether current (the conserved energy-momentum density vector) to be spacelike or past directed (timelike or null) on some part of the horizon. In the particle case, our general criterion for the occurrence of a Penrose process reproduces the standard result. In the case of relativistic jet-producing "magnetically arrested disks," we show that the negative energy and angular-momentum absorption condition is obeyed when the Blandford-Znajek mechanism is at work, and hence the high energy extraction efficiency up to ~300% found in recent numerical simulations of such accretion flows results from tapping the black hole's rotational energy through the Penrose process. We show how black-hole rotational energy extraction works in this case by describing the Penrose process in terms of the Noether current.

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I. INTRODUCTION

Relativistic jets are often launched from the vicinity of accreting black holes. They are observed to be produced in stellar-mass black-hole binary systems and are believed to be the fundamental part of the gamma-ray burst phenomenon. Powerful relativistic jets are also ejected by accreting supermassive black holes in some active galactic nuclei (AGN). There is no doubt that the jet-launching mechanism is related to accretion onto black holes, but there has been no general agreement as to the ultimate source of energy of these spectacular high energy phenomena. In principle, relativistic jets can be powered either by the black hole's gravitational pull or by its rotation (spin), with large-scale magnetic fields invoked as energy extractors in both cases. Black-hole rotational energy extraction due to weakly magnetized accretion was considered by Ruffni and Wilson [1] (see also [2]). In the context of strongly magnetized jets, Blandford and Znajek [3] (BZ) proposed a model of electromagnetic extraction of a black hole's rotational energy based on the analogy with the classical Faraday disk (unipolar induction) phenomenon. The difficulty with applying this analogy to a rotating black hole was a viable identification of the analogue of the Faraday disk in a setup where the surface of the rotating body (the black hole's surface) is causally disconnected from the rest of the Universe. It seems now that this problem has been clarified and solved ([4,5] and references therein). Another subject of discussion about the physical meaning of the BZ mechanism was its relation to the black-hole rotational energy extraction process proposed by Penrose [6], in which an infalling particle decays into two in the ergoregion, with one of the decay products being absorbed by the black hole and the other one reaching infinity, with energy larger than that of the initial, infalling parent particle (see [7] for a review). The energy gain in this ("mechanical") Penrose process is explained by the negative ("seen" from infinity) energy of the ergoregion-trapped particle absorbed by the

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black hole. In the BZ mechanism, particle inertia can be neglected; therefore, it clearly is not a mechanical Penrose process. Komissarov [5] argues that the BZ mechanism is an example of an *energy counterflow*, a black-hole extraction phenomenon supposed to be more general than the Penrose process.

In the present article, we discuss the relation between any mechanism extracting black-hole rotational energy and the mechanical Penrose process using a general-relativistic, covariant description of the energy fluxes in the metric of a stationary and axisymmetric rotating black hole (this framework encompasses the Kerr metric as the special case of a black hole surrounded by non-self-gravitating matter). In particular, using energy and angular-momentum conservation laws, we prove that for *any* matter or field, tapping the black-hole rotational energy is possible if and only if negative energy and angular momentum are absorbed by the black hole and no torque at the black-hole horizon is necessary (or possible). The conditions on energy and angular-momentum fluxes through the horizon are analogous to those on particle energy and angular momentum in the mechanical Penrose process. From these conditions, we deduce a necessary condition for a general (passive) electromagnetic field configuration to allow black-hole energy extraction through the Penrose process. In the case of stationary, axisymmetric, and force-free fields, we obtain the well-known condition [3] on the angular speed of the field lines. We also describe the Penrose process in terms of the Noether current. This description is particularly useful in the description of results of numerical simulations.

Finally, we use our generalized Penrose process framework to interpret the results of recent numerical studies of accretion onto black holes by [8–10], which indicate that the BZ mechanism can tap the black-hole rotational energy very efficiently (efficiency $\eta > 100\%$). These simulations are based on large-scale numerical simulations involving a particular state of accretion around rotating black holes: "magnetically arrested disks" (MADs), first in Newtonian gravity (see, e.g., [11,12]), and later in GR (e.g., [8,9]). MADs were also called "magnetically choked accretion flows" in [10]. We show that the resulting configurations satisfy the Penrose-process conditions for black-hole energy extraction.

Our results agree, in most respects, with those obtained by Komissarov [5]. The difference between the two approaches worth noticing is that we derive our generalized Penrose condition from the fundamental, and universally accepted, *null energy condition*, while Komissarov introduces a new concept of the *energy counterflow*. This difference will be investigated in a future paper.

More than 30 years ago, Carter [4], analyzing the BZ mechanism in a covariant framework, obtained several results similar to ours. Using energy and angular-momentum *rates* (integrated fluxes, while we use energy and angular momentum) he showed the necessity of a negative energy absorption rate at the horizon for this mechanism to operate. Strangely, his paper has almost never been cited in the context of the discussion of the Penrose-BZ process. Our treatment is more general than that of Carter, since we use a general energy-momentum tensor, while Carter considered fields that are time periodic (cf. Sec. 6.4.2 of Ref. [13]). Moreover, we obtain a new condition on a general electromagnetic field configuration [Eq. (7.7) below], and we apply it to interpret recent numerical simulation of relativistic jet production.

In a recent paper [14], the MAD simulations have been described in the framework of the so-called "membrane paradigm" [15]. This picture of the interaction of electromagnetic fields with the black-hole surface has the advantage of using the analogues of the usual electric and magnetic fields in a 3-D flat space. Penna *et al.* [14] showed that the results of MAD simulations can be consistently described in the membrane framework.

II. THE MECHANICAL PENROSE PROCESS

Penrose [6] considered[1] a free-falling particle that enters the ergosphere of a rotating black hole with energy $E_1 = -\vec{\eta}\cdot\vec{p}_1$, where $\vec{\eta}$ is the Killing vector associated with stationarity [see also Eq. (3.1) below], $\vec{p}_1$ the particle 4-momentum vector, and the dot denotes the spacetime scalar product: $\vec{\eta}\cdot\vec{p}_1 = \boldsymbol{g}(\vec{\eta},\vec{p}_1) = g_{\mu\nu}\eta^\mu p_1^\nu = \eta_\mu p_1^\mu$. Here $\boldsymbol{g}$ is the metric tensor, whose signature is chosen to be $(-,+,+,+)$. Note that although E_1 is called an *energy*, it is not the particle's energy measured by any observer since $\vec{\eta}$ is not a unit vector (i.e., it cannot be considered as the 4-velocity of any observer), except in the asymptotically flat region infinitely far from the black hole. For this reason E_1 is often called the *energy at infinity*. The virtue of E_1 is to remain constant along the particle's worldline, as long as the latter is a geodesic, i.e., as long as the particle is free falling. In the ergoregion, the particle disintegrates into two particles with, say, 4-momenta $\vec{p}_2$ and $\vec{p}_*$. Their conserved energies are, respectively, $E_2 = -\vec{\eta}\cdot\vec{p}_2$ and $\Delta E_H = -\vec{\eta}\cdot\vec{p}_*$ (the notation ΔE_H is for future convenience). The first particle escapes to infinity, which implies $E_2 > 0$, while the second one falls into the black hole. Since in the ergoregion $\vec{\eta}$ is a spacelike vector (from the very definition of an ergoregion), it is possible to have $\Delta E_H < 0$ on certain geodesics. The falling particle is then called a *negative energy particle*, although its energy measured by any observer, such as for instance a zero-angular-momentum observer (ZAMO), remains always positive. At the disintegration point, the conservation of 4-momentum implies $\vec{p}_1 = \vec{p}_2 + \vec{p}_*$; taking the scalar product with $\vec{\eta}$, we deduce that $E_1 = E_2 + \Delta E_H$. Then, as a result of $\Delta E_H < 0$, we get $E_2 > E_1$. At infinity, where the constants E_1 and E_2 can be

[1] See also [17].

interpreted as the energies measured by an inertial observer at rest with respect to the black hole (thanks to the asymptotic behavior of $\vec{\eta}$), one has clearly some energy gain: the outgoing particle is more energetic than the ingoing one. This is the so-called *mechanical Penrose process* of energy extraction from a rotating black hole. In other words, the sufficient and necessary condition for energy extraction from a rotating black hole is

$$\Delta E_H < 0. \tag{2.1}$$

From the condition that energy measured locally by a ZAMO must be non-negative, one obtains (see, e.g., [16])

$$\omega_H \Delta J_H \leq \Delta E_H, \tag{2.2}$$

where ω_H is the angular velocity of the black hole (defined below) and ΔJ_H is the angular momentum of the negative-energy particle absorbed by the black hole, defined by $\Delta J_H = \vec{\xi} \cdot \vec{p}_*$, where $\vec{\xi}$ is the Killing vector associated with axisymmetry. Without loss of generality, we take $\omega_H \geq 0$. Equations (2.1)–(2.2) imply that $\omega_H \neq 0$ and

$$\Delta J_H < 0. \tag{2.3}$$

It worth stressing that in the mechanical Penrose process, particles move on geodesics along which (by construction) energy is conserved. Therefore, the negative energy particle must originate in the ergoregion, the only domain of spacetime where such a particle can exist. In the general case of interacting matter or fields, negative energy at the horizon does not imply negative energy elsewhere.

Soon after Penrose's discovery that rotating black holes may be energy sources, it was suggested that the mechanical Penrose process may power relativistic jets observed in quasars. However, a careful analysis by [18–21] (see also [7]), showed that it is unlikely that negative energy states, necessary for the Penrose process to work, may be achieved through the particles' disintegration and/or collision inside the ergosphere. This conclusion has been confirmed more recently by [22–24] for high energy particle collisions. The reason is that in the case of collisions, the particles with positive energies cannot escape because they must have large but negative radial momenta. Thus, they are captured (together with the negative energy particles) by the black hole. Note that for charged particles evolving in the electromagnetic field of a Kerr-Newman black hole, the efficiency of the mechanical Penrose process can be very large [7,25].

Attempts to describe the BZ mechanism as a mechanical Penrose process have been unsuccessful ([5] and references therein). This leaves electromagnetic processes as the only astrophysically realistic way to extract rotational energy from a rotating black hole.

III. GENERAL RELATIVISTIC PRELIMINARIES

A. The spacetime symmetries

The spacetime is modeled by a four-dimensional smooth manifold $\mathcal{M}$ equipped with a metric $\boldsymbol{g}$ of signature $(-,+,+,+)$. We are considering a rotating uncharged black hole that is stationary and axisymmetric. If the black hole is isolated, i.e., not surrounded by self-gravitating matter or electromagnetic fields, the spacetime $(\mathcal{M}, \boldsymbol{g})$ is described by the Kerr metric (see Appendix A). Here and in Secs. IV to VII, we do not restrict to this case and consider a generic stationary and axisymmetric metric $\boldsymbol{g}$. As already mentioned in Sec. II, we denote by $\vec{\eta}$ the Killing vector associated with stationarity and by $\vec{\xi}$ that associated with axisymmetry. In a coordinate system $(x^\alpha) = (t, x^1, x^2, x^3)$ adapted to stationarity, i.e., such that

$$\frac{\partial}{\partial t} = \vec{\eta}, \tag{3.1}$$

the components $g_{\alpha\beta}$ of the metric tensor are independent of the coordinate t. In a similar way, if the coordinate x^3, say, corresponds to the axial symmetry, the components $g_{\alpha\beta}$ will be independent of this coordinate.

B. The black-hole horizon

The event horizon $\mathcal{H}$ is a null hypersurface; if it is stationary and axisymmetric, the symmetry generators $\vec{\eta}$ and $\vec{\xi}$ have to be tangent to it (cf. Fig. 1). Moreover, any null normal $\vec{\ell}$ to $\mathcal{H}$ has to be a linear combination of $\vec{\eta}$ and $\vec{\xi}$: up to some rescaling by a constant factor, we may write

$$\vec{\ell} = \vec{\eta} + \omega_H \vec{\xi}, \tag{3.2}$$

where $\omega_H \geq 0$ is constant over $\mathcal{H}$ (rigidity theorem, cf. [13]) and is called the *black-hole angular velocity*. Since ω_H is constant, $\vec{\ell}$ is itself a Killing vector and $\mathcal{H}$ is called a *Killing horizon*. For a Kerr black hole of mass m and angular momentum am, we have $\omega_H = a/[2mr_H]$, where $r_H = m + \sqrt{m^2 - a^2}$ is the radius of the black-hole horizon. Since $\mathcal{H}$ is a null hypersurface, the normal $\vec{\ell}$ is null, $\vec{\ell} \cdot \vec{\ell} = 0$. For this reason, $\vec{\ell}$ is both normal and tangent to $\mathcal{H}$. The field lines of $\vec{\ell}$ are *null geodesics* tangent to $\mathcal{H}$; they are called the *null generators* of $\mathcal{H}$. One of them is drawn in Fig. 1.

Let $(x^\alpha) = (t, x^1, x^2, x^3)$ be a coordinate system on $\mathcal{M}$ that is adapted to the stationarity, in the sense of (3.1), and regular on $\mathcal{H}$. In the case of a Kerr black hole, this means that (x^α) are not the standard Boyer-Lindquist coordinates, which are well known to be singular on $\mathcal{H}$. Regular coordinates on $\mathcal{H}$ are the Kerr coordinates, either in their original version [26] or in the $3+1$ one, and the Kerr-Schild coordinates, which are used in the numerical

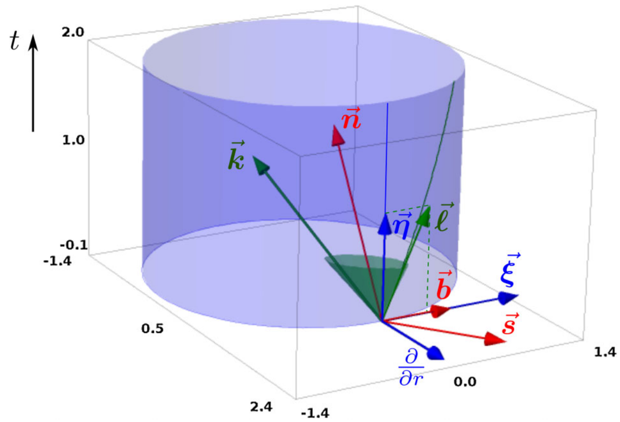

FIG. 1 (color online). Spacetime diagram showing the event horizon of a Kerr black hole of angular-momentum parameter $a/m = 0.9$. This three-dimensional diagram is cut at $\theta = \pi/2$ of the four-dimensional spacetime. The diagram is based on the 3 + 1 Kerr coordinates (t, r, ϕ) described in Appendix A and the axes are labeled in units of m. The event horizon $\mathcal{H}$ is the blue cylinder of radius $r = r_H = 1.435\,m$ [this value results from $a = 0.9\,m$ via (A3)] and the green cone is the future light cone at the point $(t = 0, \theta = \pi/2, \varphi = 0)$ on $\mathcal{H}$. The null vectors $\vec{\ell}$ and $\vec{k}$ (drawn in green) are tangent to this light cone, but not $\vec{\eta}$ which, although tangent to $\mathcal{H}$, being spacelike lies outside of the light cone. Note that relation (3.2) holds with $\omega_H = 0.313\ m^{-1}$ (cf. Appendix A). The green line, to which $\vec{\ell}$ is tangent, is a null geodesic tangent to $\mathcal{H}$; if the figure was extended upward, it would show up as a helix. $\vec{n}$ is the (timelike) unit normal to the hypersurface $t = 0$. $\vec{s}$ is the (spacelike) unit normal to the 2-sphere $\mathcal{S}_0$ defined by $t = 0$ and $r = r_H$. Note that this 2-sphere is drawn here as a circle (the basis of the cylinder) because the dimension along θ has been suppressed. The vector $\vec{b}$ is the unit vector along $\vec{\xi} = \partial/\partial\varphi$. The vectors $(\vec{n}, \vec{s}, \vec{b})$ form an orthonormal basis (drawn in red) for the metric $\boldsymbol{g}$.

computations by Tchekhovskoy et al. [8,27] and McKinney et al. [10], discussed in Sec. VIII. See Appendix A for more details on the coordinate system and the coordinate representation of $\vec{\ell}$.

Then from (3.1) and (3.2), t is the parameter along the null geodesics generating $\mathcal{H}$ for which $\vec{\ell}$ is the tangent vector:

$$\ell^\alpha = \frac{\mathrm{d}x^\alpha}{\mathrm{d}t}. \tag{3.3}$$

(Note that in general t is not an affine parameter along these geodesics.) Since the coordinates (t, x^i) are assumed regular on $\mathcal{H}$, the 2-surfaces $\mathcal{S}_t$ of constant t on $\mathcal{H}$ provide a regular slicing of $\mathcal{H}$ by a family of spacelike 2-spheres. Let us denote by $\vec{k}$ the future-directed null vector field defined on $\mathcal{H}$ by the following requirements (cf. Fig. 1):

(1) $\vec{k}$ is orthogonal to $\mathcal{S}_t$,
(2) $\vec{k}$ obeys

$$\vec{k} \cdot \vec{\ell} = -1. \tag{3.4}$$

Then, at each point of $\mathcal{S}_t$, Span$(\vec{k}, \vec{\ell})$ is the timelike 2-plane orthogonal to $\mathcal{S}_t$. Note that $\vec{k}$ is transverse to $\mathcal{H}$ (i.e., is not tangent to it) and that, contrary to $\vec{\ell}$, the vector $\vec{k}$ depends on the choice of the coordinates (t, x^i) (more precisely on the slicing $(\mathcal{S}_t)_{t\in\mathbb{R}}$ of $\mathcal{H}$; see, e.g., [28]).

The 2-surfaces $\mathcal{S}_t$ of constant t on $\mathcal{H}$ are spacelike 2-spheres corresponding to what is commonly understood as the "black-hole surface," in analogy to the "stellar surface."

C. Energy condition

Let $\boldsymbol{T}$ be the energy-momentum tensor of matter and nongravitational fields surrounding the black hole. We shall assume that it fulfills the so-called *null energy condition* at the event horizon:

$$T_{\mu\nu}\ell^\mu\ell^\nu|_{\mathcal{H}} \geq 0. \tag{3.5}$$

This is a very mild condition, which is satisfied by any ordinary matter and any electromagnetic field. In particular, it follows (by some continuity argument timelike → null) from the standard *weak energy condition* [29], according to which energy measured locally by observers is always non-negative.

IV. ENERGY AND ANGULAR-MOMENTUM CONSERVATION LAWS

In the mechanical Penrose process, particles move on geodesics along which the energy E and the angular momentum J, as defined in Sec. II, are conserved quantities. Therefore, they can be evaluated anywhere along the particle trajectories, in particular at the black-hole surface where an energy flux can be calculated. In the general case of matter with nongravitational interactions (e.g., a perfect fluid) or a field (e.g., electromagnetic), the energy and angular momentum must be evaluated using the conservation equations, and in such a case the fluxes of the conserved quantities play the role equivalent to that of energy and angular momentum in the case of particles.[2]

A. Energy conservation

Let us consider the "energy-momentum density" vector $\vec{P}$ defined by

$$P^\alpha = -T^\alpha{}_\mu \eta^\mu. \tag{4.1}$$

If matter and nongravitational fields obey the standard *dominant energy condition* [29], then $\vec{P}$ must be a future-directed timelike or null vector as long as $\vec{\eta}$ is

[2]In Abramowicz et al. [30] where generalizing the Penrose process was attempted, Eqs. (B3) and (B4) are not correct because the "energy at infinity" and "angular momentum at infinity" that are used there are not conserved quantities.

timelike, i.e., outside the ergoregion. In the ergoregion, where $\vec{\eta}$ is spacelike, there is no guarantee that $\vec{P}$ is timelike or null, and even when it is timelike, $\vec{P}$ can be past directed (an example is provided in Fig. 5 below). Therefore, $\vec{P}$ cannot be interpreted as a physical energy-momentum density, hence the quotes in the above denomination. Moreover, even outside the ergoregion, $\vec{P}$ does not correspond to the energy-momentum density measured by any physical observer, since $\vec{\eta}$ fails to be some observer's 4-velocity, not being a unit vector, except at infinity (cf. the discussion in Sec. II). The vector $\vec{P}$ is known as the *Noether current* associated with the symmetry generator $\vec{\eta}$ [31,32]. It is conserved in the sense that

$$\nabla_\mu P^\mu = 0. \tag{4.2}$$

This is easily proved from the definition (4.1) by means of (i) the energy-momentum conservation law $\nabla_\mu T^{\mu\nu} = 0$, (ii) the Killing equation obeyed by $\vec{\eta}$, and (iii) the symmetry of the tensor $\boldsymbol{T}$. By Stokes's theorem, it follows from (4.2) that the flux of $\vec{P}$ through any closed[3] oriented hypersurface $\mathcal{V}$ vanishes:

$$\oint_{\mathcal{V}} \boldsymbol{\epsilon}(\vec{P}) = 0, \tag{4.3}$$

where $\boldsymbol{\epsilon}(\vec{P})$ stands for the 3-form obtained by setting $\vec{P}$ as the first argument of the Levi-Civita tensor $\boldsymbol{\epsilon}$ (or volume 4-form) associated with the spacetime metric $\boldsymbol{g}$:

$$\boldsymbol{\epsilon}(\vec{P}) \coloneqq \boldsymbol{\epsilon}(\vec{P}, ., ., .). \tag{4.4}$$

In terms of components in a right-handed basis,

$$\epsilon(\vec{P})_{\alpha\beta\gamma} = P^\mu \epsilon_{\mu\alpha\beta\gamma} = \sqrt{-g} P^\mu [\mu, \alpha, \beta, \gamma], \tag{4.5}$$

where $g \coloneqq \det(g_{\alpha\beta})$ and $[\mu, \alpha, \beta, \gamma]$ is the alternating symbol of four indices; i.e., $[\mu, \alpha, \beta, \gamma] = 1$ (-1) if $(\mu, \alpha, \beta, \gamma)$ is an even (odd) permutation of (0, 1, 2, 3), and $[\mu, \alpha, \beta, \gamma] = 0$ otherwise. Note that the integral (4.3) is intrinsically well defined, as the integral of a 3-form over a three-dimensional oriented manifold. The proof of (4.3) relies on Stokes's theorem according to which the integral over $\mathcal{V}$ is equal to the integral over the interior of $\mathcal{V}$ of the exterior derivative of the 3-form $\boldsymbol{\epsilon}(\vec{P})$; the latter being $\mathbf{d}[\boldsymbol{\epsilon}(\vec{P})] = (\nabla_\mu P^\mu)\boldsymbol{\epsilon}$, it vanishes identically as a consequence of (4.2).

Let us apply (4.3) to the hypersurface $\mathcal{V}$ defined as the following union:

$$\mathcal{V} \coloneqq \Sigma_1 \cup \Delta\mathcal{H} \cup \Sigma_2 \cup \Sigma_{\rm ext}, \tag{4.6}$$

where (cf. Fig. 2)

[3] In other words, it is compact without boundary.

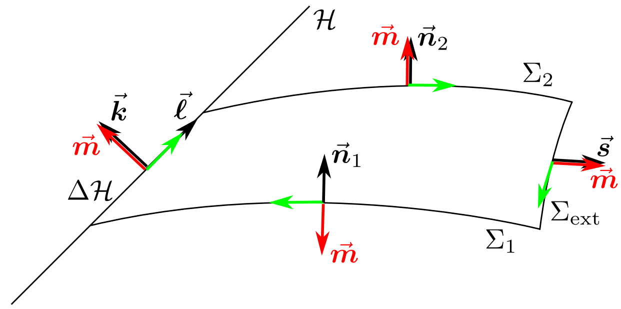

FIG. 2 (color online). Closed hypersurface $\mathcal{V} = \Sigma_1 \cup \Delta\mathcal{H} \cup \Sigma_2 \cup \Sigma_{\rm ext}$. The green arrows depict the orientation of $\mathcal{V}$, which is given by $\boldsymbol{\epsilon}(\vec{m})$.

(i) Σ_1 (Σ_2) is a compact spacelike hypersurface delimited by two 2-spheres, $\mathcal{S}_1$ and $\mathcal{S}_1^{\rm ext}$ ($\mathcal{S}_2$ and $\mathcal{S}_2^{\rm ext}$), such that $\mathcal{S}_1$ ($\mathcal{S}_2$) lies on $\mathcal{H}$ and $\mathcal{S}_1^{\rm ext}$ ($\mathcal{S}_2^{\rm ext}$) is located far from the black hole;
(ii) Σ_2 is assumed to lie entirely in the future of Σ_1;
(iii) $\Delta\mathcal{H}$ is the portion of the event horizon $\mathcal{H}$ delimited by $\mathcal{S}_1$ and $\mathcal{S}_2$;
(iv) $\Sigma_{\rm ext}$ is a timelike hypersurface having $\mathcal{S}_1^{\rm ext}$ and $\mathcal{S}_2^{\rm ext}$ for boundaries.

We may choose (but this is not mandatory) the 2-spheres $\mathcal{S}_1$ and $\mathcal{S}_2$ to coincide with some slices of the foliation $(\mathcal{S}_t)_{t\in\mathbb{R}}$ of $\mathcal{H}$ mentioned in Sec. IIIB: $\mathcal{S}_1 = \mathcal{S}_{t_1}$ and $\mathcal{S}_2 = \mathcal{S}_{t_2}$.

We choose the orientation of $\mathcal{V}$ to be towards its exterior, but the final results do not depend upon this choice. The orientation of $\mathcal{V}$ is depicted by the vector $\vec{m}$ in Fig. 2. Note that this vector does not have to be normal to the various parts of $\mathcal{V}$ (in particular, it is not normal to $\Delta\mathcal{H}$). Its role is only to indicate that the orientation of $\mathcal{V}$ is given by the 3-form $\boldsymbol{\epsilon}(\vec{m})$ restricted to vectors tangent to $\mathcal{V}$. More precisely, $\vec{m}$ is defined as follows:

(i) on Σ_1, $\vec{m} = -\vec{n}_1$, the vector $\vec{n}_1$ being the future-directed unit timelike normal to Σ_1;
(ii) on Σ_2, $\vec{m} = \vec{n}_2$, the future-directed unit timelike normal to Σ_2;
(iii) on $\Sigma_{\rm ext}$, $\vec{m} = \vec{s}$, the unit spacelike normal to $\Sigma_{\rm ext}$ oriented towards the exterior of $\mathcal{V}$;
(iv) on $\Delta\mathcal{H}$, $\vec{m} = \vec{k}$, the future-directed null vector introduced above [cf. (3.4)].

In view of (4.6), the property (4.3) gives

$$\int_{\Sigma_1\downarrow} \boldsymbol{\epsilon}(\vec{P}) + \int_{\underset{\longleftarrow}{\Delta\mathcal{H}}} \boldsymbol{\epsilon}(\vec{P}) + \int_{\Sigma_2\uparrow} \boldsymbol{\epsilon}(\vec{P}) + \int_{\underset{\rightarrow}{\Sigma_{\rm ext}}} \boldsymbol{\epsilon}(\vec{P}) = 0, \tag{4.7}$$

where the arrows indicate the orientation (cf. Fig. 2). Let us then define the *energy contained* in Σ_1 by

$$E_1 \coloneqq \int_{\Sigma_1\uparrow} \boldsymbol{\epsilon}(\vec{P}) = -\int_{\Sigma_1} P_\mu n_1^\mu \mathrm{d}V = \int_{\Sigma_1} T_{\mu\nu}\eta^\mu n_1^\nu \sqrt{\gamma}\mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3, \tag{4.8}$$

the *energy contained* in Σ_2 by

$$E_2 := \int_{\Sigma_2\uparrow} \boldsymbol{\epsilon}(\vec{\boldsymbol{P}}) = -\int_{\Sigma_2} P_\mu n_2^\mu \mathrm{d}V = \int_{\Sigma_2} T_{\mu\nu}\eta^\mu n_2^\nu \sqrt{\gamma}\mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3, \tag{4.9}$$

the *energy captured by the black hole between* Σ_1 *and* Σ_2 by

$$\Delta E_H := \int_{\underleftarrow{\Delta\mathcal{H}}} \boldsymbol{\epsilon}(\vec{\boldsymbol{P}}) = -\int_{\Delta\mathcal{H}} P_\mu \ell^\mu \mathrm{d} = \int_{\Delta\mathcal{H}} T_{\mu\nu}\eta^\mu \ell^\nu \sqrt{q}\mathrm{d}t\mathrm{d}y^1\mathrm{d}y^2, \tag{4.10}$$

and the *energy evacuated from the system between* Σ_1 *and* Σ_2 by

$$\Delta E_{\mathrm{ext}} := \int_{\underrightarrow{\Sigma_{\mathrm{ext}}}} \boldsymbol{\epsilon}(\vec{\boldsymbol{P}}) = \int_{\Sigma_{\mathrm{ext}}} P_\mu s^\mu \mathrm{d}V = -\int_{\Sigma_{\mathrm{ext}}} T_{\mu\nu}\eta^\mu s^\nu \sqrt{-h}\mathrm{d}t\mathrm{d}y^1\mathrm{d}y^2. \tag{4.11}$$

In the above formulas,

(i) $\mathrm{d}V$ is the volume element induced on each hypersurface by the spacetime Levi-Civita tensor $\boldsymbol{\epsilon}$;

(ii) (x^1, x^2, x^3) are generic coordinates on Σ_1 and Σ_2 that are right handed with respect to the hypersurface orientation;

(iii) γ is the determinant of the components with respect to the coordinates (x^1, x^2, x^3) of the 3-metric γ induced by $\boldsymbol{g}$ on Σ_1 or Σ_2;

(iv) (t, y^1, y^2) are generic right-handed coordinates on Σ_{ext};

(v) h is the determinant of the components with respect to the coordinates (t, y^1, y^2) of the 3-metric $\boldsymbol{h}$ induced by $\boldsymbol{g}$ on Σ_{ext} ($h < 0$ since Σ_{ext} is timelike);

(vi) (t, y^1, y^2) are right-handed coordinates on $\Delta\mathcal{H}$ such that t is the parameter along the null geodesics generating $\mathcal{H}$ associated with the null normal $\vec{\ell}$ [cf. (3.3)];

(vii) q is the determinant with respect to the coordinates (y^1, y^2) of the 2-metric induced by $\boldsymbol{g}$ on the 2-surfaces $t = \mathrm{const}$ in $\Delta\mathcal{H}$.

The second and third equalities in each of Eqs. (4.8)–(4.11) are established in Appendix B.

With the above definitions, (4.7) can be written as the energy conservation law

$$E_2 + \Delta E_{\mathrm{ext}} - E_1 = -\Delta E_H. \tag{4.12}$$

Notice that the minus sign in front of E_1 arises from the change of orientation of Σ_1 between (4.7) and the definition (4.8) of E_1.

B. Angular-momentum conservation

In a way similar to (4.1), we define the angular-momentum density vector by

$$M^\alpha = T^\alpha{}_\mu \xi^\mu. \tag{4.13}$$

Since $\vec{\boldsymbol{\xi}}$ is a Killing vector, $\vec{\boldsymbol{M}}$ obeys the conservation law

$$\nabla_\mu M^\mu = 0. \tag{4.14}$$

Let us introduce the *angular momentum contained in* Σ_1 and that *contained in* Σ_2 by

$$J_1 := \int_{\Sigma_1\uparrow} \boldsymbol{\epsilon}(\vec{\boldsymbol{M}}) = -\int_{\Sigma_1} M_\mu n_1^\mu \mathrm{d}V = -\int_{\Sigma_1} T_{\mu\nu}\xi^\mu n_1^\nu \sqrt{\gamma}\mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3 \tag{4.15}$$

and

$$J_2 := \int_{\Sigma_2\uparrow} \boldsymbol{\epsilon}(\vec{\boldsymbol{M}}) = -\int_{\Sigma_2} M_\mu n_2^\mu \mathrm{d}V = -\int_{\Sigma_2} T_{\mu\nu}\xi^\mu n_2^\nu \sqrt{\gamma}\mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3, \tag{4.16}$$

the *angular momentum captured by the black hole between* Σ_1 *and* Σ_2 by

$$\Delta J_H := \int_{\underleftarrow{\Delta\mathcal{H}}} \boldsymbol{\epsilon}(\vec{\boldsymbol{M}}) = -\int_{\Delta\mathcal{H}} M_\mu \ell^\mu \mathrm{d}V = -\int_{\Delta\mathcal{H}} T_{\mu\nu}\xi^\mu \ell^\nu \sqrt{q}\mathrm{d}t\mathrm{d}y^1\mathrm{d}y^2, \tag{4.17}$$

and the *angular momentum evacuated from the system between* Σ_1 and Σ_2 by

$$J_{\mathrm{ext}} := \int_{\underrightarrow{\Sigma_{\mathrm{ext}}}} \boldsymbol{\epsilon}(\vec{\boldsymbol{M}}) = \int_{\Sigma_{\mathrm{ext}}} M_\mu s^\mu \mathrm{d}V = \int_{\Sigma_{\mathrm{ext}}} T_{\mu\nu}\xi^\mu s^\nu \sqrt{-h}\mathrm{d}t\mathrm{d}y^1\mathrm{d}y^2. \tag{4.18}$$

We deduce then from (4.14) that, similarly to (4.12),

$$J_2 + J_{\mathrm{ext}} - J_1 = -\Delta J_H. \tag{4.19}$$

C. Explicit expressions in adapted coordinates

Let us call *adapted coordinates* any right-handed spherical-type coordinate system $(x^\alpha) = (t, r, \theta, \varphi)$ such that (i) t and φ are associated with the two spacetime symmetries, so that the two independent Killing vectors are $\vec{\boldsymbol{\eta}} = \partial/\partial t$ and $\vec{\boldsymbol{\xi}} = \partial/\partial\varphi$; (ii) the event horizon $\mathcal{H}$ is the hypersurface defined by $r = \mathrm{const} = r_H$; (iii) the timelike hypersurface Σ_{ext} is defined by $r = \mathrm{const} = r_{\mathrm{ext}}$ and

$t \in [t_1, t_2]$, where t_1 and t_2 are two constants such that $t_1 < t_2$; and (iv) the spacelike hypersurface Σ_1 (Σ_2) is defined by $t = t_1$ and $r \in [r_H, r_{\rm ext}]$ ($t = t_2$ and $r \in [r_H, r_{\rm ext}]$). Then $\Delta\mathcal{H}$ is the hypersurface defined by $r = r_H$ and $t \in [t_1, t_2]$. In the case of Kerr spacetime, an example of adapted coordinates is the $3+1$ Kerr coordinates described in Appendix A.

On Σ_1 or Σ_2, (r, θ, φ) are coordinates that are right handed with respect to the "up" orientation of these hypersurfaces used in the definitions (4.8)–(4.9) of E_1 and E_2. Consequently,

$$E_{1,2} = \int_{\Sigma_{1,2}} \epsilon(P)_{r\theta\varphi} \mathrm{d}r\mathrm{d}\theta\mathrm{d}\varphi = \int_{\Sigma_{1,2}} \sqrt{-g} P^t \underbrace{[t, r, \theta, \varphi]}_{1} \mathrm{d}r\mathrm{d}\theta\mathrm{d}\varphi,$$

where the second equality results from (4.5). Now, (4.1) yields $P^t = -T^t{}_t \eta^\mu = -T^t{}_t$ since $\eta^\alpha = (1, 0, 0, 0)$ in adapted coordinates. We conclude that

$$E_1 = -\int_{\Sigma_1} T^t{}_t \sqrt{-g} \mathrm{d}r\mathrm{d}\theta\mathrm{d}\varphi \quad \text{and}$$
$$E_2 = -\int_{\Sigma_2} T^t{}_t \sqrt{-g} \mathrm{d}r\mathrm{d}\theta\mathrm{d}\varphi. \tag{4.20}$$

As a check, we note that the above formulas can also be recovered from the expressions involving $T_{\mu\nu}\eta^\mu n^\nu_{1,2}$ in (4.8)–(4.9). Indeed, the unit timelike normal $\vec{n}$ to Σ_1 or Σ_2 obeys $n_\alpha = (-N, 0, 0, 0)$, where N is the lapse function of the spacetime foliation by $t =$ const hypersurfaces (see, e.g., [33]). Accordingly $T_{\mu\nu}\eta^\mu n^\nu = T^\nu{}_\mu \eta^\mu n_\nu = T^t{}_t(-N)$. Since $N\sqrt{\gamma} = \sqrt{-g}$, we get (4.20).

On $\Delta\mathcal{H}$, (t, θ, φ) are coordinates that are right handed with respect to the "inward" orientation used in the definition (4.10) of ΔE_H. Indeed

$$\begin{aligned}\boldsymbol{\epsilon}(\vec{\boldsymbol{m}}, \vec{\boldsymbol{\partial}}_t, \vec{\boldsymbol{\partial}}_\theta, \vec{\boldsymbol{\partial}}_\varphi) &= \boldsymbol{\epsilon}(\vec{\boldsymbol{k}}, \vec{\boldsymbol{\partial}}_t, \vec{\boldsymbol{\partial}}_\theta, \vec{\boldsymbol{\partial}}_\varphi) \\ &= k^r \epsilon_{rt\theta\varphi} = -\underbrace{k^r}_{<0} \underbrace{\epsilon_{tr\theta\varphi}}_{>0} > 0.\end{aligned} \tag{4.21}$$

Accordingly,

$$\Delta E_H = \int_{\Delta\mathcal{H}} \epsilon(P)_{t\theta\varphi} \mathrm{d}t\mathrm{d}\theta\mathrm{d}\varphi = \int_{\Delta\mathcal{H}} \sqrt{-g} P^r \underbrace{[r, t, \theta, \varphi]}_{-1} \mathrm{d}t\mathrm{d}\theta\mathrm{d}\varphi, \tag{4.22}$$

where the second equality results from (4.5). Since $P^r = -T^r{}_t$ from (4.1), we get

$$\Delta E_H = \int_{\Delta\mathcal{H}} T^r{}_t \sqrt{-g} \mathrm{d}t\mathrm{d}\theta\mathrm{d}\varphi. \tag{4.23}$$

On $\Sigma_{\rm ext}$, it is (t, φ, θ), and not (t, θ, φ), that constitutes a right-handed coordinate system with respect to the orientation used in the definition (4.11) of $\Delta E_{\rm ext}$. Indeed

$$\begin{aligned}\boldsymbol{\epsilon}(\vec{\boldsymbol{m}}, \vec{\boldsymbol{\partial}}_t, \vec{\boldsymbol{\partial}}_\varphi, \vec{\boldsymbol{\partial}}_\theta) &= \boldsymbol{\epsilon}(\vec{\boldsymbol{s}}, \vec{\boldsymbol{\partial}}_t, \vec{\boldsymbol{\partial}}_\varphi, \vec{\boldsymbol{\partial}}_\theta) \\ &= s^r \epsilon_{rt\varphi\theta} = \underbrace{s^r}_{>0} \underbrace{\epsilon_{tr\theta\varphi}}_{>0} > 0.\end{aligned} \tag{4.24}$$

We have, therefore,

$$\Delta E_{\rm ext} = \int_{\Sigma_{\rm ext}} \epsilon(P)_{t\varphi\theta} \mathrm{d}t\mathrm{d}\theta\mathrm{d}\varphi = \int_{\Sigma_{\rm ext}} \sqrt{-g} P^r \underbrace{[r, t, \varphi, \theta]}_{1} \mathrm{d}t\mathrm{d}\theta\mathrm{d}\varphi. \tag{4.25}$$

Substituting $-T^r{}_t$ for P^r, we get

$$\Delta E_{\rm ext} = -\int_{\Sigma_{\rm ext}} T^r{}_t \sqrt{-g} \mathrm{d}t\mathrm{d}\theta\mathrm{d}\varphi. \tag{4.26}$$

The formulas for the angular momentum are similar to the above ones, with $T^t{}_t$ replaced by $-T^t{}_\varphi$ and $T^r{}_t$ replaced by $-T^r{}_\varphi$:

$$J_1 = \int_{\Sigma_1} T^t{}_\varphi \sqrt{-g} \mathrm{d}r\mathrm{d}\theta\mathrm{d}\varphi \quad \text{and}$$
$$J_2 = \int_{\Sigma_2} T^t{}_\varphi \sqrt{-g} \mathrm{d}r\mathrm{d}\theta\mathrm{d}\varphi, \tag{4.27}$$

$$\Delta J_H = -\int_{\Delta\mathcal{H}} T^r{}_\varphi \sqrt{-g} \mathrm{d}t\mathrm{d}\theta\mathrm{d}\varphi, \tag{4.28}$$

$$\Delta J_{\rm ext} = \int_{\Sigma_{\rm ext}} T^r{}_\varphi \sqrt{-g} \mathrm{d}t\mathrm{d}\theta\mathrm{d}\varphi. \tag{4.29}$$

Expressions (4.20)–(4.26) and (4.27)–(4.29), as well as the energy conservation law (4.12) and the angular-momentum conservation law (4.19), are rederived in Appendix D, via a pure coordinate-based calculation.

V. GENERAL CONDITIONS FOR BLACK-HOLE ROTATIONAL ENERGY EXTRACTION

A. General case

For definiteness, let us consider that Σ_1 and Σ_2 are parts of a foliation of spacetime by a family of spacelike hypersurfaces $(\Sigma_t)_{t\in\mathbb{R}}$:

$$\Sigma_1 = \Sigma_{t_1} \quad \text{and} \quad \Sigma_2 = \Sigma_{t_2} \quad \text{with} \quad t_2 > t_1. \tag{5.1}$$

For instance, in the case of a Kerr black hole, the hypersurface label t can be chosen to be the Kerr-Schild time coordinate introduced in Appendix A.

In (4.12), we may then interpret E_1 as the "initial energy," i.e., the energy "at the time t_1"; E_2 as the "final energy," i.e., the energy "at the time t_2"; and $\Delta E_{\rm ext}$ as the energy evacuated from the system between the times t_1 and t_2. Accordingly, the "energy gained by the world outside of the black hole" between t_1 and t_2 is defined as

$$\Delta E := E_2 + \Delta E_{\rm ext} - E_1. \tag{5.2}$$

Then, energy will be extracted from the black hole if and only if $\Delta E > 0$. In view of the conservation law (4.12), we conclude that energy is extracted from a black hole if and only if

$$\Delta E_H < 0. \tag{5.3}$$

We refer to any process that accomplishes this as a *Penrose process*.

Let us assume that the energy-momentum tensor obeys the *null energy condition* (cf. Sec. IIIC) on the event horizon: $T_{\mu\nu}\ell^\mu\ell^\nu|_{\mathcal{H}} \geq 0$ [Eq. (3.5)]. As mentioned above, this is a rather mild condition, implied by the standard weak energy condition. From (3.2), (4.1), and (4.13), it follows that

$$T_{\mu\nu}\ell^\mu\ell^\nu = T_{\mu\nu}(\eta^\nu + \omega_H\xi^\nu)\ell^\mu = -P_\mu\ell^\mu + \omega_H M_\mu\ell^\mu.$$

Integrating (3.5) over $\Delta\mathcal{H}$ yields then

$$-\int_{\Delta\mathcal{H}} P_\mu\ell^\mu \mathrm{d}V + \omega_H \int_{\Delta\mathcal{H}} M_\mu\ell^\mu \mathrm{d}V \geq 0, \tag{5.4}$$

where we have used the fact that ω_H is constant. Using (4.10) and (4.17), the above relation can be rewritten as $\Delta E_H - \omega_H\Delta J_H \geq 0$, i.e.,

$$\omega_H\Delta J_H \leq \Delta E_H. \tag{5.5}$$

In view of (5.5) and $\omega_H \geq 0$, the black-hole energy extraction condition (5.3) implies

$$\Delta J_H < 0. \tag{5.6}$$

We conclude the following:

> *For a matter distribution or a nongravitational field obeying the null energy condition, a necessary and sufficient condition for energy extraction from a rotating black hole is that it absorbs negative energy ΔE_H and negative angular momentum ΔJ_H.*

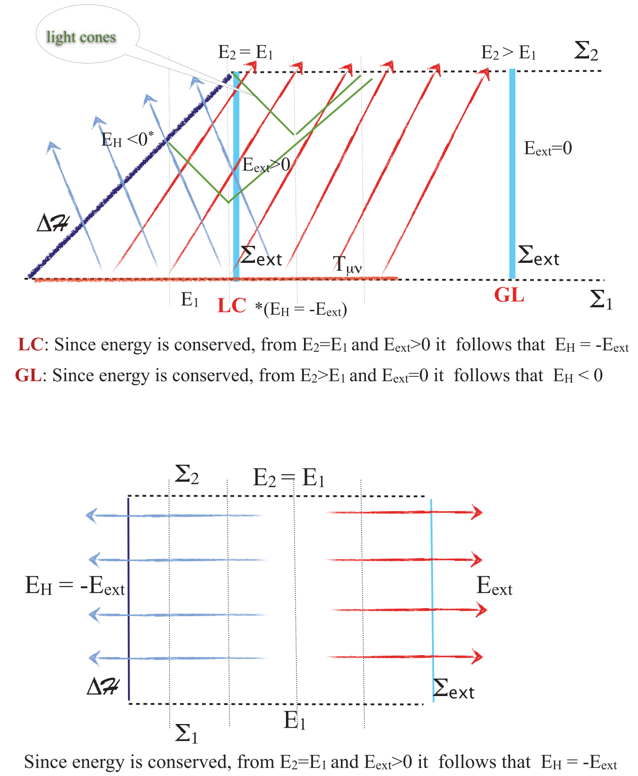

FIG. 3 (color online). Two views of the energy balance in a Penrose process. Top: Global (GL) with $E_2 > E_1$ and $\Delta E_{\rm ext} = 0$. Bottom: local (LC) stationary view with $E_2 = E_1$ but $\Delta E_{\rm ext} = -\Delta E_H > 0$. The region of spacetime concerned with this view is marked "LC" on the top figure.

Equations (5.3), (5.5), and (5.6) are identical with Eqs. (2.1), (2.2), and (2.3), describing the condition for the Penrose process. They describe the same physics: in order to extract energy from a rotating black hole, one must feed it negative energy and angular momentum.

Any extraction of a black hole's rotational energy by interaction with matter and/or (nongravitational) fields is a Penrose process.

B. Penrose process in terms of the Noether current $\vec{P}$

Given the expression (4.10) of ΔE_H, we note that the Penrose-process condition (5.3) implies $P_\mu\ell^\mu > 0$ on some part of $\Delta\mathcal{H}$. Since $\vec{\ell}$ is a future-directed null vector, $P_\mu\ell^\mu > 0$ if and only if $\vec{P}$ is either (i) spacelike or (ii) past directed timelike or past directed null. Therefore, we conclude that

> *A necessary condition for a Penrose process to occur is to have the Noether current $\vec{P}$ be spacelike or past directed (timelike or null) on some part of $\Delta\mathcal{H}$.*

As we already noticed in Sec. IVA, if the matter or fields fulfil the standard dominant energy condition, the vector $\vec{P}$ is always future directed timelike or null outside the ergoregion; therefore, it can be spacelike or past directed only in the ergoregion.

C. Applications of the Penrose-process energy balance

The energy balance equations derived above can be applied to basically two views of energy extraction from a black hole. First, one can use the global (GL) spacetime view applied to theoretically described "real" astrophysical systems (Fig. 3, top). Matter and/or fields have limited space extent, and the timelike hypersurface Σ_{ext} is placed sufficiently far so that $\Delta E_{\mathrm{ext}} = 0$. When there is energy extraction, i.e., when $\Delta E > 0$, then $E_2 > E_1$. This is the view we will have in mind in Secs. VI and VII.

When dealing with numerical simulations, however, such global view is usually unpractical. The simulation is performed in a box of limited size and the system is brought to a stationary state. The view presented in the bottom part of Fig. 3 is then more adapted to the energy balance. Because of stationarity, one has $E_2 = E_1$ but $\Delta E_{\mathrm{ext}} > 0$. When the numerical code conserves energy very well, the energy balance implies $\Delta E_H < 0$. This is the view applied in Sec. VIII.

VI. VARIOUS EXAMPLES OF THE PENROSE PROCESS

In what follows we will apply Eqs. (4.8) to (4.12) and (4.15) to (4.19) to various black-hole plus matter (or fields) configurations. We first show that in the case of particles, one recovers the standard Penrose-process formulas. Then we shall apply our formalism to the cases of a scalar field and a perfect fluid. The case of the electromagnetic field is treated in Sec. VII.

A. Mechanical Penrose-process test

Let us show that the formalism developed above reproduces the mechanical Penrose process for a single particle that breaks up into two fragments in the ergoregion.

The energy-momentum tensor of a massive particle of mass $\mathfrak{m}$ and 4-velocity $\vec{u}$ is (cf. e.g., [34])

$$T_{\alpha\beta}(M) = \mathfrak{m}\int_{-\infty}^{+\infty}\delta_{A(\tau)}(M)g_\alpha{}^\mu(M,A(\tau))u_\mu(\tau)\times g_\beta{}^\nu(M,A(\tau))u_\nu(\tau)\mathrm{d}\tau, \tag{6.1}$$

where $M\in\mathcal{M}$ is the spacetime point at which $T_{\alpha\beta}$ is evaluated, τ stands for the particle's proper time, $A(\tau)\in\mathcal{M}$ is the spacetime point occupied by the particle at the proper time τ, $g_\alpha{}^\mu(M,A)$ is the parallel propagator from the point A to the point M along the unique geodesic[4] connecting A to M (cf. Sec. 5 of [34] or Appendix I of [35]), and $\delta_A(M)$ is the Dirac distribution on $(\mathcal{M},\boldsymbol{g})$ centered at the point A: it is defined by the identity

[4]Thanks to the Dirac distribution in (6.1), only the limit $M\to A$ matters, so that we can assume that there is a unique geodesic connecting A to M.

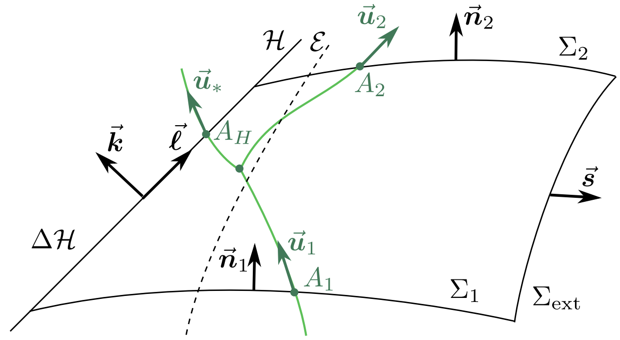

FIG. 4 (color online). Penrose process for a particle. The dashed line $\mathcal{E}$ marks the ergosphere.

$$\int_{\mathcal{U}}\delta_A(M)f(M)\sqrt{-g}\mathrm{d}^4x = f(A), \tag{6.2}$$

for any four-dimensional domain $\mathcal{U}$ around A and any scalar field $f\colon\mathcal{U}\to\mathbb{R}$. In terms of a coordinate system (x^α) around A,

$$\delta_A(M) = \frac{1}{\sqrt{-g}}\delta(x^0-z^0)\delta(x^1-z^1)\delta(x^2-z^2)\delta(x^3-z^3), \tag{6.3}$$

where δ is the standard Dirac distribution on $\mathbb{R}$, (x^α) are the coordinates of M, (z^α) those of A, and g is the determinant of the components of the metric tensor with respect to the coordinates (x^α).

The Noether current corresponding to (6.1) is formed via (4.1):

$$P_\alpha(M) = \mathfrak{m}\int_{-\infty}^{+\infty}\delta_{A(\tau)}(M)[-g_\sigma{}^\nu(M,A(\tau))u_\nu(\tau)\eta^\sigma(M)]\times g_\alpha{}^\mu(M,A(\tau))u_\mu(\tau)\mathrm{d}\tau. \tag{6.4}$$

This means that $\vec{P}$ is a distribution vector whose support is the particle's worldline and that is collinear to the particle's 4-velocity.

Let us choose Σ_1 and Σ_2 such that Σ_1 encounters the original particle $\mathcal{P}_1$ (mass $\mathfrak{m}_1$, 4-velocity $\vec{u}_1$) at the event A_1, Σ_2 encounters the escaping fragment $\mathcal{P}_2$ (mass $\mathfrak{m}_2$, 4-velocity $\vec{u}_2$) at the event A_2, and the infalling fragment $\mathcal{P}_*$ (mass $\mathfrak{m}_*$, 4-velocity $\vec{u}_*$) crosses the horizon on $\Delta\mathcal{H}$, at the event A_H (cf. Fig. 4). By plugging (6.1) into (4.8), we get

$$E_1 = \mathfrak{m}_1\int_{\Sigma_1}\int_{-\infty}^{+\infty}\delta_{A(\tau)}(M)g_\mu{}^\rho(M,A(\tau))(u_1)_\rho(\tau)\times g_\nu{}^\sigma(M,A(\tau))(u_1)_\sigma(\tau)\eta^\mu(M)n_1^\nu(M)\times\sqrt{\gamma}\mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3\mathrm{d}\tau. \tag{6.5}$$

This formula (see Appendix C1) can be reduced to

$$E_1 = -\mathfrak{m}_1\left(\eta_\mu u_1^\mu\right)\big|_{A_1} = -\mathfrak{m}_1\eta_\mu u_1^\mu, \tag{6.6}$$

where the second equality stems from the fact that $\eta_\mu u_1^\mu$ is constant along $\mathcal{P}_1$'s worldline, since the latter is a geodesic and $\vec{\boldsymbol{\eta}}$ is a Killing vector. That $\mathcal{P}_1$'s worldline is a geodesic follows from the energy-momentum conservation law $\nabla_\mu T^{\alpha\mu} = 0$ with the form (6.1) for the energy-momentum tensor (see Sec. 19.1 of [34] for details). We recover in (6.6) the standard expression of the energy involved in textbook discussions of the Penrose process (see [16,35,36] and Sec. II).

Similarly, for the outgoing particle one gets

$$E_2 = -\mathfrak{m}_2 \eta_\mu u_2^\mu. \tag{6.7}$$

For the particle crossing the horizon, by plugging (6.1) with the characteristics of the infalling fragment $\mathcal{P}_*$ into (4.10), we get

$$\Delta E_H = \mathfrak{m}_* \int_{\Delta\mathcal{H}} \int_{-\infty}^{\infty} \delta_{A(\tau)}(M)(u_*)_\mu(\tau)\eta^\mu(M)(u_*)_\nu(\tau) \times \ell^\nu(M)\sqrt{q}\mathrm{d}t\mathrm{d}y^1\mathrm{d}y^2\mathrm{d}\tau. \tag{6.8}$$

As shown in Appendix C2, this can be reduced to

$$\Delta E_H = -\mathfrak{m}_* \left(\eta_\mu u_*^\mu\right)\big|_{A_H} = -\mathfrak{m}_* \eta_\mu u_*^\mu. \tag{6.9}$$

As for $\mathcal{P}_1$ and $\mathcal{P}_2$, the independence of $\eta_\mu u_*^\mu$ from the specific point of $\mathcal{P}_*$'s worldline where it is evaluated results from the fact that $\mathcal{P}_*$'s worldline is a geodesic.

Finally, in the present case, we have clearly $\Delta E_{\mathrm{ext}} = 0$. Therefore, the energy gain formula (5.2) reduces to $\Delta E = E_2 - E_1$, and we recover the standard Penrose process discussed in Sec. II: $E_2 > E_1$ if and only if $\Delta E_H < 0$, i.e., if and only if $\eta_\mu u_*^\mu > 0$. This is possible only in the ergoregion, where the Killing vector $\vec{\boldsymbol{\eta}}$ is spacelike. Note that $\eta_\mu u_*^\mu > 0$ implies that the term in square brackets in (6.4) is negative, so that the Noether current $\vec{\boldsymbol{P}}_*$ of $\mathcal{P}_*$ is a timelike vector (being collinear to $\vec{\boldsymbol{u}}_*$) that is past directed. This is in agreement with the statement made in Sec. VB and is illustrated in Fig. 5.

B. Scalar field (super-radiance)

Let us consider a complex scalar field Φ ruled by the standard Lagrangian

$$\mathcal{L} = -\frac{1}{2}[\nabla_\mu\bar{\Phi}\nabla^\mu\Phi + V(|\Phi|^2)], \tag{6.10}$$

where $\bar{\Phi}$ stands for Φ's complex conjugate and $V(|\Phi|^2)$ is some potential $[V(|\Phi|^2) = (\mathfrak{m}/\hbar)^2|\Phi|^2$ for a free field of mass $\mathfrak{m}]$. The corresponding energy-momentum tensor is

$$T_{\alpha\beta} = \nabla_{(\alpha}\bar{\Phi}\nabla_{\beta)}\Phi - \frac{1}{2}[\nabla_\mu\bar{\Phi}\nabla^\mu\Phi + V(|\Phi|^2)]g_{\alpha\beta}. \tag{6.11}$$

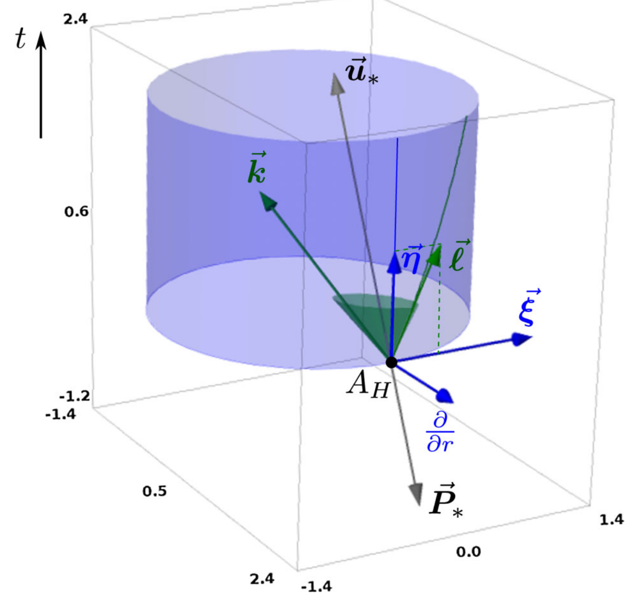

FIG. 5 (color online). Spacetime diagram showing the 4-velocity $\vec{\boldsymbol{u}}_*$ and the energy-momentum density vector $\vec{\boldsymbol{P}}_*$ of a negative energy particle $\mathcal{P}_*$ entering the event horizon of a Kerr black hole of angular-momentum parameter $a/m = 0.9$ (see Figs. 1 and 4). At the horizon, the particle is characterized by the following coordinate velocity: $\mathrm{d}r/\mathrm{d}t = -0.32$, $\mathrm{d}\theta/\mathrm{d}t = 0$, and $\mathrm{d}\varphi/\mathrm{d}t = -0.18\omega_H$, resulting in the 4-velocity $u_*^\alpha = (2.38, -0.76, 0, -0.13)$ and in the positive scalar product $\eta_\mu u_*^\mu = 0.042$. The "vector" $\vec{\boldsymbol{P}}_*$, which is actually a distribution, is drawn with an arbitrary scale.

Let us plug the above expression into (4.10); using adapted coordinates (t, r, θ, φ) (cf. Sec. IVC), we have $\eta^\mu\nabla_\mu\Phi = \partial\Phi/\partial t$ and $\ell^\mu\nabla_\mu\Phi = \partial\Phi/\partial t + \omega_H\partial\Phi/\partial\varphi$. In addition, $g_{\mu\nu}\eta^\mu\ell^\nu = 0$, since $\vec{\boldsymbol{\eta}}$ is tangent to $\mathcal{H}$ and $\vec{\boldsymbol{\ell}}$ is the normal to $\mathcal{H}$ (cf. Sec. IIIB). Therefore, we get

$$\Delta E_H = \int_{\Delta\mathcal{H}} \left[\frac{\partial\Phi}{\partial t}\frac{\partial\bar{\Phi}}{\partial t} + \frac{\omega_H}{2}\left(\frac{\partial\Phi}{\partial t}\frac{\partial\bar{\Phi}}{\partial\varphi} + \frac{\partial\bar{\Phi}}{\partial t}\frac{\partial\Phi}{\partial\varphi}\right)\right] \times \sqrt{q}\mathrm{d}t\mathrm{d}\theta\mathrm{d}\varphi. \tag{6.12}$$

Let us consider a rotating scalar field of the form

$$\Phi(t, r, \theta, \varphi) = \Phi_0(r, \theta)e^{i(\omega t - m\varphi)}, \tag{6.13}$$

where $\Phi_0(r, \theta)$ is a real-valued function, ω is a constant, and m some integer. Then, (6.12) becomes

$$\Delta E_H = \int_{\Delta\mathcal{H}} \Phi_0^2\omega(\omega - m\omega_H)\sqrt{q}\mathrm{d}t\mathrm{d}\theta\mathrm{d}\varphi. \tag{6.14}$$

In view of (5.3), we deduce immediately that a necessary and sufficient condition for a Penrose process to occur is

$$0 < \omega < m\omega_H. \tag{6.15}$$

In this context, the Penrose process is called *superradiance* (see, e.g., [36] and [37]). Condition (6.15) was obtained by Carter [13] in the more general case of a (not necessarily scalar) tensor field that is periodic in t with period $2\pi/\omega$.

C. Perfect fluid

Let us now consider a perfect fluid of 4-velocity $\vec{u}$, proper energy density ε and pressure p. The corresponding energy-momentum tensor is

$$T_{\alpha\beta} = (\varepsilon + p)u_\alpha u_\beta + p g_{\alpha\beta}. \tag{6.16}$$

Accordingly, and using $g_{\mu\nu}\eta^\mu \ell^\nu = 0$ as in Sec. VIB, formula (4.10) becomes

$$\Delta E_H = \int_{\Delta\mathcal{H}} (\varepsilon + p)\eta_\mu u^\mu \ell_\nu u^\nu \sqrt{q}\mathrm{d}t\mathrm{d}y^1\mathrm{d}y^2. \tag{6.17}$$

$\vec{\ell}$ being a future-directed null vector and $\vec{u}$ a future-directed timelike vector, we have necessarily

$$\ell_\nu u^\nu < 0. \tag{6.18}$$

According to (5.3), the Penrose process takes place if and only if $\Delta E_H < 0$. From (6.17), (6.18), and the assumption $\varepsilon + p \geq 0$ (the weak energy condition), we conclude that for a perfect fluid, a necessary condition for the Penrose process to occur is

$$\eta_\mu u^\mu > 0 \text{ in some part of } \Delta\mathcal{H}. \tag{6.19}$$

We may have $\eta_\mu u^\mu > 0$ in some part of $\Delta\mathcal{H}$ only because $\vec{\eta}$ is there a spacelike vector (for $\mathcal{H}$ is inside the ergoregion). Note that (6.18) and (3.2) imply

$$\omega_H \xi_\mu u^\mu < -\eta_\mu u^\mu. \tag{6.20}$$

Hence, in the parts of $\Delta\mathcal{H}$ where $\eta_\mu u^\mu > 0$, we have $\xi_\mu u^\mu < 0$. Therefore for a perfect fluid, a necessary condition for the Penrose process to occur is

$$\xi_\mu u^\mu < 0 \text{ in some part of } \Delta\mathcal{H}. \tag{6.21}$$

In other words, the fluid flow must have some azimuthal component counterrotating with respect to the black hole in some part of $\Delta\mathcal{H}$. However, no physical process extracting black-hole rotational energy through interaction with a perfect fluid is known.

In the special case of dust (fluid with $p = 0$), the fluid lines are geodesics and we recover from (6.19) the single-particle condition $\Delta E_H < 0$, with ΔE_H given by (6.9).

VII. ELECTROMAGNETIC FIELDS

A. General electromagnetic field

Let us consider some electromagnetic field, described by the field 2-form $\boldsymbol{F}$. For the moment we will deal with the most general case, i.e., that $\boldsymbol{F}$ is not necessarily stationary or axisymmetric. Of course this is possible only if $\boldsymbol{F}$ is a passive field, i.e., does not contribute as a source to the Einstein equation, so that the spacetime metric remains stationary and axisymmetric.

The electromagnetic energy-momentum tensor is given by the standard formula:

$$T_{\alpha\beta} = \frac{1}{\mu_0}\left(F_{\mu\alpha}F^\mu{}_\beta - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}g_{\alpha\beta}\right). \tag{7.1}$$

Accordingly, the integrand in formula (4.10) for ΔE_H is

$$\boldsymbol{T}(\vec{\eta}, \vec{\ell}) = \frac{1}{\mu_0}\left(F_{\mu\rho}\eta^\rho F^\mu{}_\sigma \ell^\sigma - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\vec{\eta}\cdot\vec{\ell}\right).$$

Now, since $\vec{\eta}$ is tangent to $\mathcal{H}$ and $\vec{\ell}$ normal to $\mathcal{H}$, one has $\vec{\eta}\cdot\vec{\ell} = 0$. There remains then

$$\mu_0 \boldsymbol{T}(\vec{\eta}, \vec{\ell}) = F_{\mu\rho}\eta^\rho F^\mu{}_\sigma \ell^\sigma. \tag{7.2}$$

Let us introduce on $\mathcal{H}$ the "pseudoelectric field" 1-form [13,38–40]

$$\boldsymbol{E} \coloneqq \boldsymbol{F}(., \vec{\ell}). \tag{7.3}$$

If $\vec{\ell}$ were a unit timelike vector, $\boldsymbol{E}$ would be a genuine electric field, namely, the electric field measured by the observer whose 4-velocity is $\vec{\ell}$. But in the present case, $\vec{\ell}$ is a null vector, so that such a physical interpretation does not hold. $\boldsymbol{E}$ is called a corotating electric field in [13,38] because $\vec{\ell}$ is the corotating Killing vector on $\mathcal{H}$. Note that,[5] thanks to the antisymmetry of $\boldsymbol{F}$,

$$\langle \boldsymbol{E}, \vec{\ell}\rangle = 0. \tag{7.4}$$

This implies that the vector $\vec{E}$ deduced from the 1-form $\boldsymbol{E}$ by metric duality (i.e., the vector of components $E^\alpha = g^{\alpha\mu}E_\mu = F^\alpha{}_\mu \ell^\mu$) is tangent to $\mathcal{H}$. Equation (7.2) can be written as

$$\mu_0 \boldsymbol{T}(\vec{\eta}, \vec{\ell}) = \boldsymbol{F}(\vec{E}, \vec{\eta}). \tag{7.5}$$

Thanks to (3.2) and (7.3), this expression can be recast as

[5] In this section, we are using index-free notations. In particular, the action of a 1-form on a vector is denoted by brackets, $\langle \boldsymbol{E}, \vec{\ell}\rangle = E_\mu \ell^\mu$, and the scalar product of two vectors is denoted with a dot, $\vec{u}\cdot\vec{v} = g_{\mu\nu}u^\mu v^\nu = u_\nu v^\nu$.

$$\mu_0 \boldsymbol{T}(\vec{\boldsymbol{\eta}},\vec{\ell}) = \boldsymbol{F}(\vec{\boldsymbol{E}},\vec{\ell}-\omega_H\vec{\boldsymbol{\xi}}) = \boldsymbol{F}(\vec{\boldsymbol{E}},\vec{\ell}) - \omega_H \boldsymbol{F}(\vec{\boldsymbol{E}},\vec{\boldsymbol{\xi}}) = \langle \boldsymbol{E},\vec{\boldsymbol{E}}\rangle - \omega_H \boldsymbol{F}(\vec{\boldsymbol{E}},\vec{\boldsymbol{\xi}}),$$

i.e.,

$$\mu_0 \boldsymbol{T}(\vec{\boldsymbol{\eta}},\vec{\ell}) = \vec{\boldsymbol{E}}\cdot\vec{\boldsymbol{E}} - \omega_H \boldsymbol{F}(\vec{\boldsymbol{E}},\vec{\boldsymbol{\xi}}). \tag{7.6}$$

Given expression (4.10) for ΔE_H, we conclude that the necessary condition for the Penrose process to occur is

$$\omega_H \boldsymbol{F}(\vec{\boldsymbol{E}},\vec{\boldsymbol{\xi}}) > \vec{\boldsymbol{E}}\cdot\vec{\boldsymbol{E}} \text{ in some part of } \Delta\mathcal{H}. \tag{7.7}$$

Note that since $\vec{\boldsymbol{E}}$ is tangent to $\mathcal{H}$ [cf. (7.4)] and $\mathcal{H}$ is a null hypersurface, $\vec{\boldsymbol{E}}$ is either a null vector or a spacelike one, so that in (7.7) one has always

$$\vec{\boldsymbol{E}}\cdot\vec{\boldsymbol{E}} \geq 0. \tag{7.8}$$

Equation (7.7) is the most general condition on any electromagnetic field configuration allowing black-hole energy extraction through a Penrose process. Obviously, for $\omega_H = 0$ there is no energy extraction.

B. Stationary and axisymmetric electromagnetic field

In this section, we assume that the electromagnetic field obeys the spacetime symmetries, which are expressed by

$$\mathcal{L}_{\vec{\boldsymbol{\eta}}}\boldsymbol{F} = 0 \quad \text{and} \quad \mathcal{L}_{\vec{\boldsymbol{\xi}}}\boldsymbol{F} = 0, \tag{7.9}$$

where $\mathcal{L}_{\vec{\boldsymbol{v}}}$ stands for the Lie derivative along the vector field $\vec{\boldsymbol{v}}$. Then it can be shown (see, e.g., [41] for details) that $\boldsymbol{F}$ is entirely determined by three scalar fields Φ, Ψ, and I, such that

$$\boldsymbol{F}(.,\vec{\boldsymbol{\eta}}) = \mathbf{d}\Phi \tag{7.10}$$

$$\boldsymbol{F}(.,\vec{\boldsymbol{\xi}}) = \mathbf{d}\Psi \tag{7.11}$$

$$^\star\boldsymbol{F}(\vec{\boldsymbol{\eta}},\vec{\boldsymbol{\xi}}) = I, \tag{7.12}$$

where $\mathbf{d}$ is the exterior derivative operator (reducing to the gradient for a scalar field such as Φ or Ψ) and $^\star\boldsymbol{F}$ stands for the Hodge dual of $\boldsymbol{F}$. Note that, being defined solely from $\boldsymbol{F}$ and the Killing fields $\vec{\boldsymbol{\eta}}$ and $\vec{\boldsymbol{\xi}}$, Φ, Ψ, and I are gauge-independent quantities. Introducing an electromagnetic potential 1-form $\boldsymbol{A}$ such that $\boldsymbol{F} = \mathbf{d}\boldsymbol{A}$, one may use the standard electromagnetic gauge freedom to choose $\boldsymbol{A}$ so that

$$\Phi = \langle \boldsymbol{A},\vec{\boldsymbol{\eta}}\rangle = A_t \quad \text{and} \quad \Psi = \langle \boldsymbol{A},\vec{\boldsymbol{\xi}}\rangle = A_\phi. \tag{7.13}$$

In addition to (7.10)–(7.12), one has (see, e.g., [41]) $\boldsymbol{F}(\vec{\boldsymbol{\eta}},\vec{\boldsymbol{\xi}}) = 0$ and

$$\mathcal{L}_{\vec{\boldsymbol{\eta}}}\Phi = \mathcal{L}_{\vec{\boldsymbol{\xi}}}\Phi = 0 \quad \text{and} \quad \mathcal{L}_{\vec{\boldsymbol{\eta}}}\Psi = \mathcal{L}_{\vec{\boldsymbol{\xi}}}\Psi = 0, \tag{7.14}$$

which means that the scalar fields Φ and Ψ obey the two spacetime symmetries.

From the definition (7.3) and expression (3.2) of $\vec{\ell}$, the corotating pseudoelectric field $\boldsymbol{E}$ is

$$\boldsymbol{E} = \boldsymbol{F}(.,\vec{\ell}) = \boldsymbol{F}(.,\vec{\boldsymbol{\eta}}) + \omega_H \boldsymbol{F}(.,\vec{\boldsymbol{\xi}}) = \mathbf{d}\Phi + \omega_H \mathbf{d}\Psi,$$

where the last equality follows from (7.10) and (7.11). Since ω_H is constant, we conclude that the 1-form $\boldsymbol{E}$ is a pure gradient:

$$\boldsymbol{E} = \mathbf{d}(\Phi + \omega_H\Psi). \tag{7.15}$$

Remark: If the electromagnetic field is not passive, i.e., if it contributes significantly to the spacetime metric via the Einstein equation, then $\boldsymbol{T}(\vec{\ell},\vec{\ell})$ must vanish in order for the black hole to be in equilibrium (otherwise it would generate some horizon expansion, via the Raychaudhuri equation; see, e.g., [38]). Since by (7.1), $\boldsymbol{T}(\vec{\ell},\vec{\ell}) = \mu_0^{-1}\vec{\boldsymbol{E}}\cdot\vec{\boldsymbol{E}}$, this implies that $\vec{\boldsymbol{E}}$ is a null vector. Being tangent to $\mathcal{H}$, the only possibility is to have $\vec{\boldsymbol{E}}$ collinear to $\vec{\ell}$: $\vec{\boldsymbol{E}} = f\vec{\ell}$. Then for any vector $\vec{\boldsymbol{v}}$ tangent to $\mathcal{H}$, one has $\vec{\boldsymbol{v}}\cdot\vec{\boldsymbol{E}} = 0$. In view of (7.15), we get the remarkable result that [38]

$$\Phi + \omega_H\Psi \text{ is constant over } \mathcal{H}. \tag{7.16}$$

Returning to the case of passive fields we notice that thanks to (7.10), the ΔE_H integrand (7.5) becomes

$$\mu_0 \boldsymbol{T}(\vec{\boldsymbol{\eta}},\vec{\ell}) = \vec{\boldsymbol{E}}\cdot\vec{\boldsymbol{\nabla}}\Phi. \tag{7.17}$$

In a similar way, from (7.11) one deduces that the ΔJ_H integrand $\mu_0 \boldsymbol{T}(\vec{\boldsymbol{\xi}},\vec{\ell}) = \boldsymbol{F}(\vec{\boldsymbol{E}},\vec{\boldsymbol{\xi}})$ takes the form of

$$\mu_0 \boldsymbol{T}(\vec{\boldsymbol{\xi}},\vec{\ell}) = \vec{\boldsymbol{E}}\cdot\vec{\boldsymbol{\nabla}}\Psi. \tag{7.18}$$

In view of (7.15), we get

$$\mu_0 \boldsymbol{T}(\vec{\boldsymbol{\eta}},\vec{\ell}) = \vec{\boldsymbol{\nabla}}\Phi\cdot\vec{\boldsymbol{\nabla}}(\Phi + \omega_H\Psi). \tag{7.19}$$

C. Force-free stationary and axisymmetric field (Blandford-Znajek)

Let us assume that the electromagnetic field is force free, in addition to being stationary and axisymmetric:

$$\boldsymbol{F}(\vec{\boldsymbol{j}},.) = 0, \tag{7.20}$$

where $\vec{\boldsymbol{j}}$ is the electric 4-current. In particular, $\boldsymbol{F}(\vec{\boldsymbol{j}},\vec{\boldsymbol{\eta}}) = 0$ and $\boldsymbol{F}(\vec{\boldsymbol{j}},\vec{\boldsymbol{\xi}}) = 0$. From (7.10) and (7.11), it follows immediately that

$$\vec{\boldsymbol{j}}\cdot\vec{\boldsymbol{\nabla}}\Phi = 0 \quad \text{and} \quad \vec{\boldsymbol{j}}\cdot\vec{\boldsymbol{\nabla}}\Psi = 0. \tag{7.21}$$

Taking into account that Φ and Ψ are stationary and axisymmetric [cf. (7.14)], we may rewrite (7.21) in a coordinate system (t,r,θ,ϕ) adapted to stationarity and axisymmetry as

$$j^r\frac{\partial\Phi}{\partial r} + j^\theta\frac{\partial\Phi}{\partial\theta} = 0 \quad \text{and} \quad j^r\frac{\partial\Psi}{\partial r} + j^\theta\frac{\partial\Psi}{\partial\theta} = 0.$$

We deduce that, generically, there exists a function $\omega = \omega(\Psi)$ such that

$$\mathbf{d}\Phi = -\omega(\Psi)\mathbf{d}\Psi. \tag{7.22}$$

Equation (7.19) becomes then

$$\mu_0\boldsymbol{T}(\vec{\boldsymbol{\eta}},\vec{\ell}) = \omega(\Psi)(\omega(\Psi)-\omega_H)\vec{\boldsymbol{\nabla}}\Psi\cdot\vec{\boldsymbol{\nabla}}\Psi. \tag{7.23}$$

Notice also that from (7.17), (7.18), and (7.22), it follows that for an axisymmetric, stationary, and force-free field,

$$\Delta E_H = \omega(\Psi)\Delta J_H. \tag{7.24}$$

Now, we have

$$\vec{\ell}\cdot\vec{\boldsymbol{\nabla}}\Psi = \vec{\boldsymbol{\eta}}\cdot\vec{\boldsymbol{\nabla}}\Psi + \omega_H\vec{\boldsymbol{\xi}}\cdot\vec{\boldsymbol{\nabla}}\Psi = \underbrace{\mathcal{L}_{\vec{\boldsymbol{\eta}}}\Psi}_{0} + \omega_H\underbrace{\mathcal{L}_{\vec{\boldsymbol{\xi}}}\Psi}_{0} = 0.$$

This means that the vector $\vec{\boldsymbol{\nabla}}\Psi$ is tangent to $\mathcal{H}$. Since the latter is a null hypersurface, it follows that $\vec{\boldsymbol{\nabla}}\Psi$ is either null or spacelike. Therefore, on $\mathcal{H}$,

$$\vec{\boldsymbol{\nabla}}\Psi\cdot\vec{\boldsymbol{\nabla}}\Psi \geq 0. \tag{7.25}$$

Accordingly, (7.23) yields

$$\boldsymbol{T}(\vec{\boldsymbol{\eta}},\vec{\ell}) < 0 \Longleftrightarrow \begin{cases} \omega(\Psi)(\omega(\Psi)-\omega_H) < 0 \\ \vec{\boldsymbol{\nabla}}\Psi\cdot\vec{\boldsymbol{\nabla}}\Psi \neq 0 \end{cases},$$

i.e.,

$$\boldsymbol{T}(\vec{\boldsymbol{\eta}},\vec{\ell}) < 0 \Longleftrightarrow \begin{cases} 0 < \omega(\Psi) < \omega_H \\ \vec{\boldsymbol{\nabla}}\Psi\cdot\vec{\boldsymbol{\nabla}}\Psi \neq 0 \end{cases}. \tag{7.26}$$

We recover the result (4.6) of Blandford and Znajek's article [3]. In view of (4.10) and (5.3), we may conclude the following:

For a stationary and axisymmetric force-free electromagnetic field, a necessary condition for the Penrose process to occur is

$$0 < \omega(\Psi) < \omega_H \text{ in some part of } \Delta\mathcal{H}. \tag{7.27}$$

In particular, for a nonrotating black hole ($\omega_H = 0$), no Penrose process can occur. The condition (7.27) can be compared to the condition (6.15) for a scalar field.

VIII. SIMULATIONS OF ELECTROMAGNETIC EXTRACTION OF BLACK-HOLE ROTATIONAL ENERGY

Until recently, the relevance of the Blanford-Znajek process to observed high energy phenomena such as relativistic jets has been hotly debated and the efficiency of this mechanism put in doubt (see, e.g., [42,43]). Providing jet production efficiencies of less than ~20%, general relativistic magnetohydrodynamics (GRMHD) simulations were not of much help in ending the controversy. Only recently a new physical setup of GRMHD simulations [8,10] produced the first clear evidence of net energy extraction by magnetized accretion onto a spinning black hole. These simulations were carried out with general relativistic MHD code HARM [44] with recent improvements [8,45].

A. The framework

The BZ efficiency can be defined as BZ power normalized by $\dot{M}c^2$:

$$\eta_{\mathrm{BZ}} = \frac{[P_{\mathrm{BZ}}]_t}{[\dot{M}]_t c^2} = \frac{\kappa}{4\pi c}[\varphi^2_{BH}]_t\left(\frac{\omega_H r_g}{c}\right)^2 f(\omega_H), \tag{8.1}$$

where $\dot{M}$ is the accretion rate; $[\ldots]_t$ designates the time average; $\kappa \approx 0.05$ depends weakly on the magnetic field geometry, $\varphi^2_{BH} = \Phi^2_{BH}/\dot{M}r^2_g c$, Φ_{BH} being the magnetic flux through the black-hole surface; $f(\omega_H) \approx 0.77$ for $a_* = 1$, where $a_* = J/m^2$ [9]; and $r_g = Gm/c^2$ is the black-hole gravitational radius.

The efficiency η_{BZ} depends on spin and the magnetic flux on the black hole. The spin is limited by $a_* < 1.0$ ($\omega_H < c/r_S$, where $r_S = 2Gm/c^2$). The magnetic flux is limited by two factors: (1) How much of it can be pushed on to the black hole, (2) How much of it can be accumulated by diffusion through the accretion flow. In an MHD turbulent disk, accumulation of dynamically important magnetic field is possible only if it is not geometrically thin, i.e., only if $h/r \sim 1$ [46]. Tchekhovskoy *et al.* [8] considered "slim" disks ($h/r \sim 0.3$) in which initially poloidal magnetic fields are accumulated at the black hole until they obstruct the accretion and lead to the formation of a so-called magnetically arrested disk [11,12]. In such a configuration $\varphi_{BH} \sim 40$ for $a_* = 0.99$, leading to

$\eta_{BZ} > 100\%$, i.e., to *net* energy extraction from a rotating black hole.

This result, as well as subsequent simulations of various MAD[6] configurations [10], leaves little doubt that the Blandford-Znajek mechanism can play a fundamental role in the launching of (at least some) relativistic jets from the vicinity of black-hole surfaces. This conclusion is supported by observational evidence of the role of spin *and* accumulated magnetic flux in the launching of relativistic jets both in microquasars and AGNs (see, e.g., [47–51]).

In the previous section, we obtained several conditions for the occurrence of a Penrose process in the presence of electromagnetic fields. All these criteria follow from the fundamental requirement $\Delta E_H < 0$. The most general criterion applies to any electromagnetic field configuration: from the definition (4.10) and the general condition (5.3), we deduced a specific (necessary) condition (7.7) for the electromagnetic fields on the horizon. We then showed that in the case of stationary and axisymmetric force-free fields, the condition (5.3) is equivalent to the Blandford and Znajek [3] condition on the angular velocity of the magnetic field lines. In this section we will apply these conditions to the results of GRMHD simulations of magnetized jets we have discussed above. The aim of this exercise is twofold. First, using rigorous general-relativistic criteria we will confirm that the MAD BZ mechanism is indeed a Penrose process, as surmised by Tchekhovskoy *et al.* [8]. Second, our Penrose-process conditions can be used as a diagnostic tool to test the physical and mathematical consistency of numerical calculations reputed to represent the Blandford-Znajek/Penrose process.

In dealing with results of numerical simulations, we will adopt the 3 + 1 Kerr coordinates (t, r, θ, φ) described in Appendix A, which are adapted coordinates in the sense defined in Sec. IVC. The energy captured by the black hole over $\Delta\mathcal{H}$ is given by (4.23). Since for the 3 + 1 Kerr coordinates, $\sqrt{-g} = (r^2 + a^2\cos^2\theta)\sin\theta$ [cf. (A4)], we get

$$\Delta E_H = \int_{\Delta\mathcal{H}} \dot{e}_H (r_H^2 + a^2\cos^2\theta)\sin\theta \mathrm{d}t\mathrm{d}\theta\mathrm{d}\varphi, \tag{8.2}$$

where we have defined

$$\dot{e}_H := -P^r|_{\mathcal{H}} = T^r{}_t|_{\mathcal{H}}. \tag{8.3}$$

As a check of (8.2), we may recover it from the last integral in Eq. (4.10), noticing that $\eta^\mu = (1, 0, 0, 0)$, $\ell_r = (r_H^2 + a^2\cos^2\theta)/(2mr_H)$, and $\sqrt{q} = 2mr_H \sin\theta$ [cf. (A11) in Appendix A].

[6]These were also called magnetically choked accretion flows by McKinney *et al.* [10].

A formula analogous to (8.2), with $\dot{e}_H$ replaced by $-T^r{}_\varphi$, gives ΔJ_H [cf. (4.28)]; accordingly, we define

$$\dot{j}_H := -M^r|_{\mathcal{H}} = -T^r{}_\varphi|_{\mathcal{H}}. \tag{8.4}$$

Since, as discussed in Sec. VC, in numerical simulations one assumes stationarity, and Σ_2 is deduced from Σ_1 by time translation, one must have $E_2 = E_1$ (see Fig. 3). Therefore, to test the Penrose-process condition (5.3) and (5.5) and show the details of the BZ mechanism, we found it convenient to use the energy and angular-momentum *flux densities* $\dot{e}_H(t, \theta, \varphi)$ and $\dot{j}_H(t, \theta, \varphi)$ defined by (8.3) and (8.4), and plot their (t- and φ-averaged) longitudinal distribution on the t-constant 2-surface $\mathcal{S}_t$ (the black hole's surface; see Sec. IIIB) on $\mathcal{H}$.

In the MAD simulations, the energy-momentum tensor is the sum of the perfect fluid (6.16) and the electromagnetic (7.1) tensors:

$$T_{\mu\nu} = T^{(\mathrm{MA})}_{\mu\nu} + T^{(\mathrm{EM})}_{\mu\nu}.$$

Consequently we define $\dot{e}_{\mathrm{MA}} := T^{(\mathrm{MA})r}{}_t$ and $\dot{j}_{\mathrm{MA}} := -T^{(\mathrm{MA})r}{}_\varphi$; $\dot{e}_{\mathrm{EM}}$ and $\dot{j}_{\mathrm{EM}}$ are defined in an analogous way through the electromagnetic energy-momentum tensor. In the simulation of force-free fields, $\dot{e}_{\mathrm{MA}} = 0$.

The pseudoelectric field (7.3) is $E_\alpha = F_{\alpha\mu}\ell^\mu$. Therefore, in the index notation, the general necessary condition (7.7) for the Penrose process to occur takes the form

$$\omega_H F_{\mu\nu} E^\mu \xi^\nu - E_\mu E^\mu|_{\mathcal{H}} > 0. \tag{8.5}$$

In the case of MAD simulations, which are intrinsically time variable, we run the simulations long enough to achieve a quasisteady state in which all quantities fluctuate about their mean values, so we use the time average of the left-hand side in (8.5).

B. Force-free stationary electromagnetic field

As a warm-up, we present the results of simulations of black-hole rotational energy extraction by a force-free electromagnetic field. As illustration, we consider the simple case of a paraboloidal magnetic field for an $a_* = 0.99$ black hole. The field configuration corresponds to the $\nu = 1$ case of Tchekhovskoy *et al.* [27], where additional information about the setup of the problem can be found. We have chosen a paraboloidal field in preference to a monopole because of the similarity of results with those of MAD simulations discussed later.

First, in Fig. 6 we present the results of testing the general condition (8.5). It is satisfied on the whole of the black-hole surface $\mathcal{S}_t$. Also (7.8) is satisfied, which confirms that the simulations correctly reproduce the spacetime structure near and at the horizon. Since condition (8.5) follows from the requirement of negative energy on the horizon, we checked the consistency of the numerical scheme by

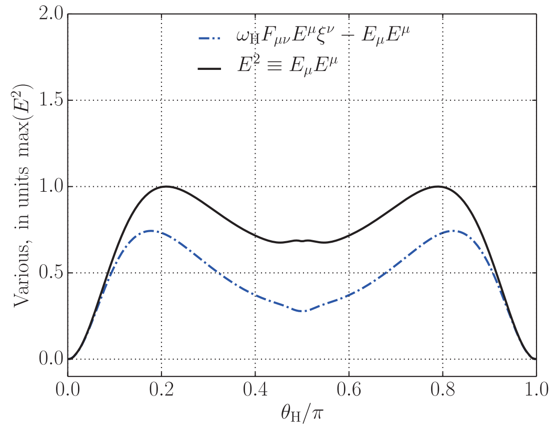

FIG. 6 (color online). Values of $\omega_H F_{\mu\nu} E^\mu \xi^\nu - E_\mu E^\mu$ and $E_\mu E^\mu$ plotted as a function of θ for a stationary, axisymmetric, and force-free field with $a_* = 0.99$. The necessary condition (8.5) for the occurrence of a Penrose process is satisfied over all $\mathcal{S}_t$ (except the poles).

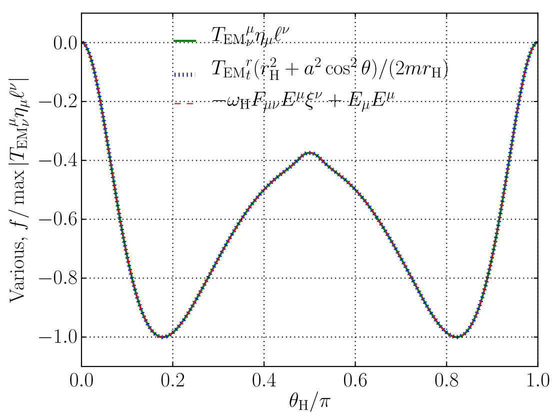

FIG. 7 (color online). Comparison of $-\omega_H F_{\mu\nu} E^\mu \xi^\nu + E_\mu E^\mu$ with the integrands of (8.2) for the same field configuration and black-hole spin as in Fig. 6.

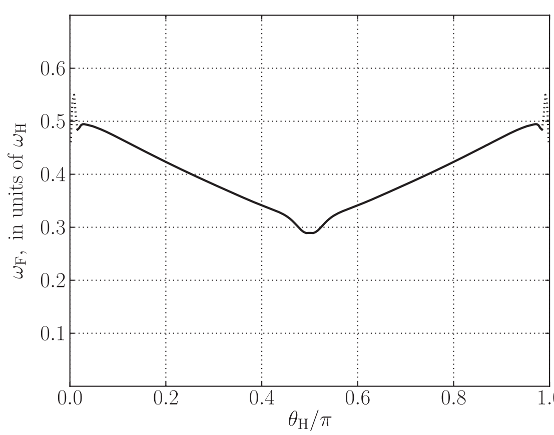

FIG. 8. ω_F/ω_H plotted as a function of θ for a stationary, axisymmetric, force-free paraboloidal magnetic field; $a_* = 0.99$. The condition (7.27) for the occurrence of a Penrose process is satisfied over the entire black-hole 2-surface $\mathcal{S}_t$.

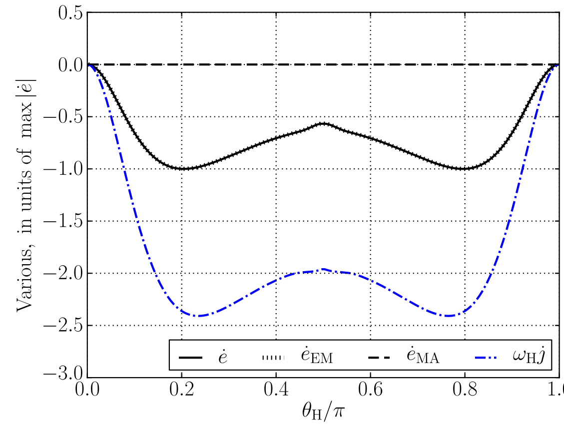

FIG. 9 (color online). Values of energy and angular-momentum density fluxes on the black-hole surface as a function of θ for a force-free field and $a_* = 0.99$. In this case, $\dot{e} = \dot{e}_{\rm EM}$. $\dot{e}$ is everywhere negative on $\mathcal{S}_t$, in agreement with the Penrose-process condition (5.3); the same is true by construction of $\dot{j}$ and (5.6) is obviously satisfied.

comparing the expression $-\omega_H F_{\mu\nu} E^\mu \xi^\nu - E_\mu E^\mu$ with two forms of the integrand in (8.2). As expected, the values of the two expressions are identical (see Fig. 7).

The force-free BZ condition (7.27) is satisfied everywhere on the black hole's surface (Fig. 8). Since in a force-free field $\dot{e}_H = \omega_H \dot{j}_H$ [cf. (7.24)], the Penrose-process condition (5.6) follows directly from $\Delta E_H < 0$ [Eq. (5.3)]; see Fig. 9.

Since it satisfies the required conditions on the horizon, the BZ mechanism described by numerical simulations of the interaction of a force-free field with a spinning black hole is a Penrose process.

C. Magnetically arrested disks

Before discussing the results of GRMHD MAD simulations in the context of the BZ/Penrose mechanism, we have first to present the underlying assumptions in more detail.

The simulations are performed in a "box" of *finite* size delimited by $\Delta\mathcal{H}$ and $\Sigma_{\rm ext}$ in space and Σ_1 and Σ_2 in time.

It is supposed that $\Sigma_{\rm ext}$ is located at some reasonably large radius ($\gtrsim 30 r_g$), which is far from the horizon but still well inside the converged volume of the simulation. One also assumes that the times t_1 and t_2 corresponding, respectively, to Σ_1 and Σ_2 are sufficiently far apart so that time averages are well defined and the system is in a steady state during this time. In a steady state, $E_2 = E_1$; i.e., the energy contained inside the volume defined by the boundaries $\Delta\mathcal{H}$ and $\Sigma_{\rm ext}$ is independent of time.

Simulation shows that $\Delta E_{\rm ext} > 0$; i.e., there is a net flow of energy out of the system. From energy conservation (4.12), one should therefore have $\Delta E_H < 0$ on some part of $\Delta\mathcal{H}$. Below we will show that stationary MAD models of energy extraction from a spinning black hole satisfy this condition and are an electromagnetic realization of a Penrose process.

We will use the results of the model A0.99N100 of McKinney *et al.* [10]. In this model, the initial magnetic field is poloidal, $a_* = 0.99$, and the disk is moderately thick: the half-thickness h satisfies $h/r \sim 0.3$ at $R_{\rm ext} = 30r_g$ and $h/r \lesssim 0.1$ at the black-hole surface.

We will first examine if the MAD simulations satisfy the Penrose-process conditions (8.5), (5.3), and (5.6). As for the force-free fields, we start with checking condition (8.5) for the electromagnetic fields on the black-hole surface. As shown in Fig. 10, $\omega_H \langle F_{\mu\nu} E^\mu \rangle \xi^\nu - \langle E_\mu E^\mu \rangle > 0$ everywhere on the black-hole surface, which implies that the electromagnetic energy is negative everywhere on $\Delta\mathcal{H}$. Indeed, as shown in Fig. 11, the electromagnetic energy density $T^{EM}_{\mu\nu}\eta^\mu \ell^\nu$ is everywhere negative on the black-hole surface. In the GRMHD MAD simulations, accretion of matter plays an essential role in accumulating magnetic field lines on the black hole, and contrary to the force-free case, the energy-momentum of matter is not negligible. In Fig. 12, in addition to the electromagnetic and matter energy density fluxes, we plot the sum of the two, representing the total energy flux. One can see that $\dot{e}$ is negative on the black-hole surface $\mathcal{S}_t$ except near the equator where energy absorption is dominated by matter accretion. Therefore, the simulations of rotational energy extraction from a $a_* = 0.99$ spinning black hole by a MAD field configuration satisfy the condition (5.3) on part of the black-hole surface and therefore describe a Penrose process involving electromagnetic fields. This is confirmed by the angular-momentum density flux being negative on the whole of the black-hole surface. We see that the angular-momentum flux is negative over the entire horizon, while the energy flux is negative only over the part of the surface exterior to the equatorial accretion flow. This is a characteristic property of the MAD configuration because the rest-mass energy flux due to the accreted mass overwhelms the energy flux into the black hole and makes it positive, while this matter carries in very little angular momentum. Its angular momentum is sub-Keplerian due to the action of strong magnetic fields that extract its angular momentum and carry it away in the form of magnetized winds.

To get more insight into the workings of the simulated black-hole rotational energy extraction process, one has to leave the horizon and see what is happening in the bulk above the black-hole surface.

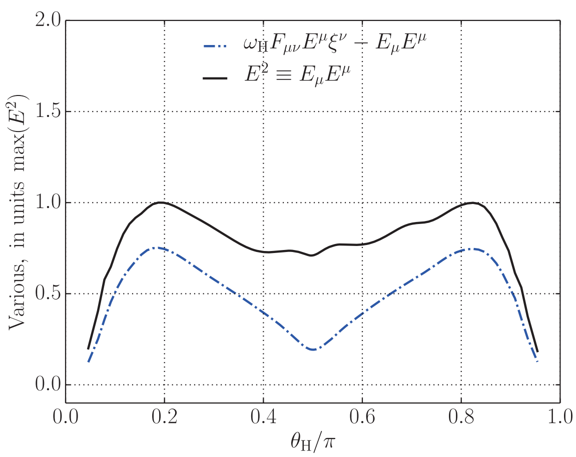

FIG. 10 (color online). Same as in Fig. 6 but for a MAD simulation with $a_* = 0.99$. Here the time- and φ-averaged quantities are used: $\omega_H \langle F_{\mu\nu} E^\mu \rangle \xi^\nu - \langle E_\mu E^\mu \rangle$ and $\langle E_\mu E^\mu \rangle$. The necessary condition (8.5) for the occurrence of a Penrose process is satisfied over all $\mathcal{S}_t$.

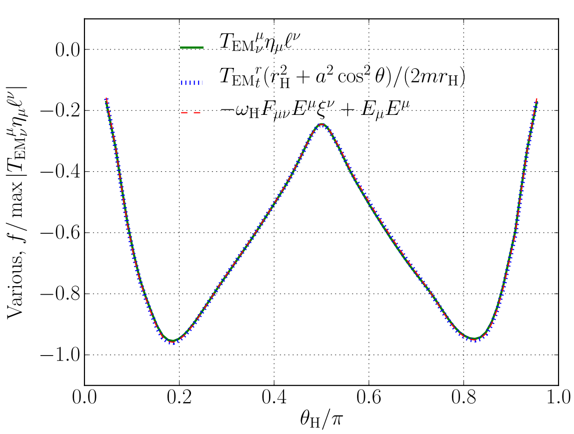

FIG. 11 (color online). Comparison of $-\omega_H \langle F_{\mu\nu} E^\mu \rangle \xi^\nu + \langle E_\mu E^\mu \rangle$ with the integrands of (8.2) for the same field configuration and black-hole spin as in Fig. 10.

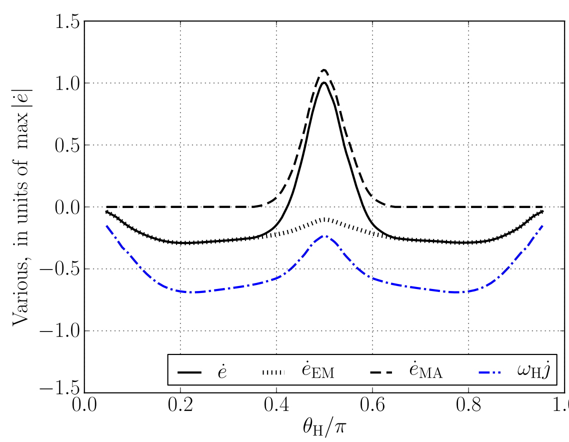

FIG. 12 (color online). Same as in Fig. 9 for a MAD configuration. The black-hole spin is $a_* = 0.99$. The electromagnetic energy density flux is everywhere negative on $\mathcal{S}_t$. The total energy density $\dot{e}$ is negative everywhere except in the equatorial belt, where matter accretion dominates the energy balance. The condition $\dot{j} < 0$ is satisfied everywhere on $\mathcal{S}_t$ (see the text for details).

We have shown that GRMHD MAD simulations of black-hole rotational energy extraction describe a Penrose process, but because of the approximations made we have not learned how this process works in detail. In the case of free particles, we know what is happening: a particle decays in the ergoregion into one with negative and another one with positive energies. The one with negative energy cannot leave the ergoregion and must be created there because negative energies exist only in the ergoregion and energy along the trajectories is conserved. This cannot be the case for a perfect fluid (with nonzero pressure) or an electromagnetic field. However, the mechanical case can serve as a guide to what is happening in a more general case. For MAD simulations, one cannot expect to see negative energies in the "bulk," since by stationarity energy is constant. However, the workings of the Penrose process should be apparent through the behavior of the Noether current $\vec{P}$. Far from the black hole, the Noether current $\vec{P}$ is future directed timelike or null and is such that positive energy flows outwards. Near the black hole, in the ergoregion, $\vec{P}$ should become spacelike or past directed. This is indeed what is happening in our simulations.

Figures 13 and 14 show the behavior of $\vec{P}$ in numerical results for the force-free and the MAD cases, respectively. We see that for a force-free configuration $P^2 = 0$ at the surface of the ergosphere, whereas in the MAD simulations the $P^2 = 0$ surface is very close to the surface of the ergosphere in the polar jet regions, but lies inside of it elsewhere. These patterns are in full agreement with Figs. 9 and 12. They demonstrate the fundamental role played by the ergoregion in extracting the black-hole energy of rotation. This can be explained analytically as follows.

In the relativistic MHD code HARM, it is assumed that the Lorentz force on a charged particle vanishes in the fluid frame:

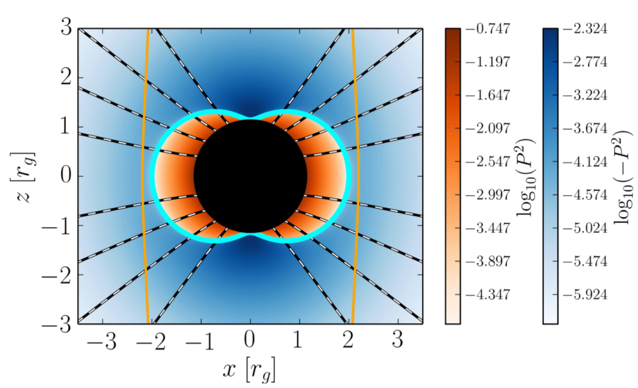

FIG. 13 (color online). Color maps of P^2 in monopolar force-free spin $a_* = 0.99$. The surface of the ergosphere is shown with cyan lines, and the stagnation surface with orange lines. The region in which $\vec{P}$ is spacelike is shown in orange, and the region in which $\vec{P}$ is timelike is shown in blue (see color bar). Black-and-white striped lines represent the magnetic field lines. As discussed in the main text, in a force-free configuration $\vec{P}$ becomes null at the surface of the ergosphere.

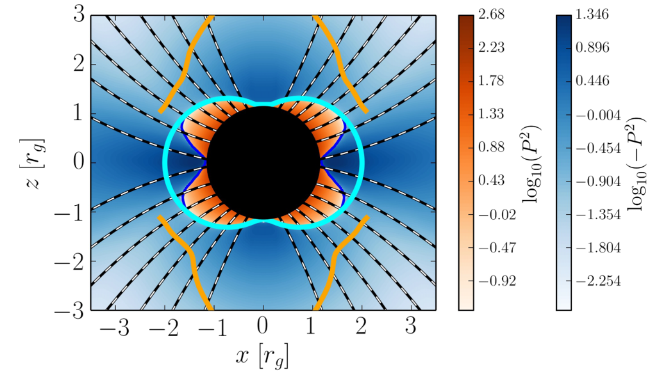

FIG. 14 (color online). Color maps of P^2 in the MAD simulations for a black hole with spin $a_* = 0.99$. Color codes and lines as in Fig. 13. In this case the surface $P^2 = 0$ nearly coincides with the surface of the ergosphere in magnetically dominated polar jets, but lies inside of the surface of the ergosphere otherwise.

$$u_\mu F^{\mu\nu} = 0. \tag{8.6}$$

Then a magnetic four-vector b^μ is defined as

$$b^\mu \coloneqq \frac{1}{2}\epsilon^{\mu\nu\alpha\beta} u_\nu F_{\alpha\beta}, \tag{8.7}$$

with

$$b_\mu u^\mu = 0, \tag{8.8}$$

following from $\boldsymbol{F}$ antisymmetry. This allows the electromagnetic energy-momentum tensor (7.1) to be written in the form of [44]:

$$T^{(\mathrm{EM})}_{\mu\nu} = b^2 u_\mu u_\nu + \frac{1}{2} b^2 g_{\mu\nu} - b_\mu b_\nu. \tag{8.9}$$

Therefore for the electromagnetic Noether current $P^{(\mathrm{EM})}_\mu = T^{(\mathrm{EM})}_{\mu\nu}\eta^\nu$, one has

$$P^\mu_{(\mathrm{EM})} P^{(\mathrm{EM})}_\mu = P^2_{(\mathrm{EM})} = \frac{1}{4} b^4 g_{tt}. \tag{8.10}$$

Since $g_{tt} > 0$ inside the ergosphere and < 0 outside, this fully explains the numerical results seen in Fig. 13:

$$P^2_{(\mathrm{EM})} > 0 \quad \text{inside ergosphere}, \tag{8.11}$$

$$P^2_{(\mathrm{EM})} < 0 \quad \text{outside ergosphere}, \tag{8.12}$$

Notice that this result applies not only to stationary axisymmetric electromagnetic force-free field but also to time-dependent fully 3D (nonaxisymmetric) configurations. However, the above property of $\vec{P}$ applies only to the electromagnetic force-free case.

To see this, let us use the general energy-momentum tensor

$$T_{\mu\nu} = T^{(MA)}_{\mu\nu} + T^{(\mathrm{EM})}_{\mu\nu},$$

with $\boldsymbol{T}^{(\mathrm{MA})}$ and $\boldsymbol{T}^{(\mathrm{EM})}$ given by (6.16) and (8.9), respectively. One obtains then

$$P^2 = \left(\frac{1}{2}b^2 + p\right)^2 g_{tt} - A, \tag{8.13}$$

with

$$A = 2(\Gamma - 1)ub_t^2 + u_t^2(\rho + u + p + b^2)[(2-\Gamma)u + \rho], \tag{8.14}$$

where $u = \epsilon - \rho$ is the internal energy and the adiabatic index Γ $[p = (\Gamma - 1)u]$ satisfies $1 \le \Gamma \le 2$ (in the MAD simulations $\Gamma = 4/3$). For dust ($p = 0$) one gets

$$P^2 = -(\rho u_t)^2;$$

i.e., the Noether current is always timelike (but past directed for negative energy worldlines; see Sec. VI A).

For the force-free case ($b^2 \gg \rho$, $p \ll \rho$), one recovers (8.10) but in general (e.g., for $\Gamma = 4/3$) $A > 0$.

Since $P^2_{\mathrm{EM}} = 0$ precisely at the surface of the ergosphere, the same applies to the full Noether current in the highly magnetized regions: there, $P^2 \approx P^2_{\mathrm{EM}} = 0$ approximately at the ergosphere. In the weakly magnetized disk-corona region, however, $P^2 = 0$ will deviate from the ergosphere by at least order unity. The first term on the right-hand side of Eq. (8.13) is positive inside the ergosphere. Since the second term is nonpositive for $1 \le \Gamma \le 2$, the surface $P^2 = 0$ lies *inside* the ergosphere, as seen in Fig. 14.

Also shown in Figs. 13 and 14 is the stagnation limit at which the field drift velocity changes sign ($u^r = 0$; inside this limit the velocity is pointing inwards). Inside the stagnation surface, an energy counterflow [5] is present: while the fields drift inwards, the energy flows outwards. The stagnation limit is always outside the ergoregion; for $a_* = 0.99$ it is very close to the ergosphere but for, e.g., $a_* = 0.9999$, the two surfaces are still well separated. The shapes and location of our stagnation limits are different from those found by Okamoto [52] and Komissarov [5]. The reasons for these differences will be addressed in a future paper.

IX. DISCUSSION AND CONCLUSIONS

We proved that for any type of matter or (nongravitational) fields satisfying the weak energy condition, the black hole's rotational energy can be extracted if and only if negative energy and angular momentum are absorbed by the black hole. Applied to the case of a single particle, the general criterion (5.3) leads to the standard condition for a mechanical Penrose process. For a general electromagnetic field, the criterion (5.3) leads to the condition (7.7) on the electromagnetic field at the horizon, which does not seem to have been expressed before.

In a sense our findings are obvious (which does not mean they are trivial). They follow from the fact that the black-hole surface is a stationary null hypersurface. Hence, it can only absorb matter or fields; it cannot emit anything, cannot emit energy. No torque can be applied to the horizon, since a torque results from a difference of material/field fluxes coming from the opposite sides of a surface [30]. The only way to lose energy, independent of the nature of the medium the hole is interacting with, is by absorbing a negative value of it. And, since the energy in question must be rotational, it must absorb negative angular momentum to slow it down.

Our results do not specify how the effect of net negative energy absorption by a black hole is achieved. The conditions for black-hole energy extraction do not guarantee the existence of such a process in the real world. As is well known, the mechanical Penrose process requires splitting of particles in the ergoregion, but no realistic way of achieving black-hole energy extraction has been found. Using fluids (perfect or not) does not seem very promising in this context. The only known black-hole energy extraction process that might be at work in the Universe is the BZ mechanism. We showed that the process of energy extraction described by GRMHD simulations of magnetically arrested disk flows around rapidly spinning black holes is a Penrose process. This has been deduced before from energy conservation and efficiencies well in excess of 100%, but we showed that the solutions found by these simulations satisfy the rigorous and general conditions required by general relativity. Considering that black holes are purely general-relativistic objects, this is a reassuring conclusion.

It is worth stressing that when in the GRMHD simulations the Noether current has a positive flux in the outward direction everywhere (including at the black-hole horizon), it does not correspond to the flow of any *physical* energy out of the black hole, since the "energy" associated with the Noether current is not a measurable quantity: no physical observer can measure it, except at infinity, where the Killing vector η becomes a unit timelike vector and therefore is eligible as the 4-velocity of a physical observer: an inertial observer at rest with respect to the black-hole location.

As mentioned above, the main (and only important) difference between the mechanical and other versions of the Penrose process is that in the first version, particles move along geodesics and therefore energy is conserved on their trajectories. Therefore, the motion of a particle crossing the horizon with negative energy is from its start restricted to the ergoregion. This does not have to be the case of interacting matter and fields. It is still true that the "outgoing flow of energy at infinity in the Penrose process is inseparable from the negative energy at infinity of an infalling 'object'" (to quote [5]), but this inseparability concerns the negative energy of the object when it is absorbed by the black hole. On its way to the final jump into the hole, the object's energy may vary depending on its interactions with the medium it is part of.

A detailed description of these processes in the framework of the GRMHD simulations will be the subject of a future work.

ACKNOWLEDGMENTS

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APPENDIX A: KERR SOLUTION IN $3+1$ KERR COORDINATES

The Kerr solution is described by two parameters: the mass m and the specific angular momentum $a \coloneqq J/m$, J being the total angular momentum. The metric components with respect to the "$3+1$" Kerr coordinates (t, r, θ, ϕ) are given by (see, e.g., [28])

$$\begin{aligned} g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = &-\left(1-\frac{2mr}{\rho^2}\right)\mathrm{d}t^2+\frac{4mr}{\rho^2}\mathrm{d}t\mathrm{d}r \\ &-\frac{4amr}{\rho^2}\sin^2\theta\mathrm{d}t\mathrm{d}\phi+\left(1+\frac{2mr}{\rho^2}\right)\mathrm{d}r^2 \\ &-2a\sin^2\theta\left(1+\frac{2mr}{\rho^2}\right)\mathrm{d}r\mathrm{d}\phi+\rho^2\mathrm{d}\theta^2 \\ &+\left(r^2+a^2+\frac{2a^2mr\sin^2\theta}{\rho^2}\right)\sin^2\theta\mathrm{d}\phi^2, \end{aligned} \tag{A1}$$

with

$$\rho^2 \coloneqq r^2 + a^2\cos^2\theta. \tag{A2}$$

The coordinates (t, r, θ, ϕ) are a $3+1$ version of the original Kerr coordinates [26] and can be viewed as a spheroidal version of the well-known "Cartesian" Kerr-Schild coordinates. The event horizon $\mathcal{H}$ is located at

$$r = r_H \coloneqq m + \sqrt{m^2 - a^2}, \tag{A3}$$

and the black-hole angular velocity ω_H defined by (3.2) takes the value $\omega_H = a/(2mr_H)$. Since r_H does not depend upon θ nor ϕ, the Kerr coordinates are adapted to $\mathcal{H}$, in the sense defined in Sec. IV C.

Note that the metric components given by Eq. (A1) are all regular at $r = r_H$.[7] Note also that in the limit $a \to 0$, then $\rho \to r$ and the line element (A1) reduces to the Schwarzschild metric in $3+1$ Eddington-Finkelstein coordinates.

From (A1), one can compute the determinant g of the metric with respect to the $3+1$ Kerr coordinates and get the relatively simple expression

$$\sqrt{-g} = (r^2 + a^2\cos^2\theta)\sin\theta. \tag{A4}$$

The metric (A1) is clearly stationary and axisymmetric, and the two vectors

$$\vec{\boldsymbol{\eta}} \coloneqq \left(\frac{\partial}{\partial t}\right)_{r,\theta,\phi} \quad \text{and} \quad \vec{\boldsymbol{\xi}} \coloneqq \left(\frac{\partial}{\partial \phi}\right)_{t,r,\theta} \tag{A5}$$

are the two Killing vectors, $\vec{\boldsymbol{\eta}}$ being associated with the stationarity and $\vec{\boldsymbol{\xi}}$ with the axial symmetry of the Kerr spacetime. These two Killing vectors are identical to the "standard" Killing vectors which are formed using the Boyer-Lindquist coordinates $(t_{\mathrm{BL}}, r, \theta, \phi_{\mathrm{BL}})$:

$$\vec{\boldsymbol{\eta}} = \left(\frac{\partial}{\partial t_{\mathrm{BL}}}\right)_{r,\theta,\phi_{\mathrm{BL}}} \quad \text{and} \quad \vec{\boldsymbol{\xi}} = \left(\frac{\partial}{\partial \phi_{\mathrm{BL}}}\right)_{t_{\mathrm{BL}},r,\theta}. \tag{A6}$$

The Killing vector $\vec{\boldsymbol{\eta}}$ ceases to be timelike at the boundary of the ergoregion (the ergosphere),

$$r_{\mathrm{erg}} = m + \sqrt{m^2 - a^2\cos^2\theta}, \tag{A7}$$

below which it is spacelike ($g_{tt} = \vec{\boldsymbol{\eta}}\cdot\vec{\boldsymbol{\eta}} > 0$).

The angular speed of the dragging of inertial frames can be written as

$$\omega = \frac{\vec{\boldsymbol{\eta}}\cdot\vec{\boldsymbol{\xi}}}{\vec{\boldsymbol{\xi}}\cdot\vec{\boldsymbol{\xi}}} = \frac{g_{t\varphi}}{g_{\varphi\varphi}} = \frac{2Jr}{A} = \frac{2amr}{A}, \tag{A8}$$

where $A = (r^2 + a^2) - \Delta a^2\sin^2\theta$ with $\Delta = r^2 - 2mr + a^2$. At the horizon $\Delta = 0$ and $\omega = \omega_H$.

Setting $\mathrm{d}r = 0$ and $r = r_H$ in the line element (A1) yields the metric γ_H induced on $\mathcal{H}$:

[7] On the contrary, most of them are singular at $\rho = 0$, which, via (A2), corresponds to $r = 0$ and $\theta = \pi/2$. In Kerr-Schild coordinates, this corresponds to the ring $x^2 + y^2 = a^2$ in the plane $z = 0$. This is the ring singularity of Kerr spacetime.

$$(\gamma_H)_{AB}\mathrm{d}x^A\mathrm{d}x^B = 2mr_H\left[(1-a\omega_H\sin^2\theta)\mathrm{d}\theta^2 + \frac{\sin^2\theta}{1-a\omega_H\sin^2\theta}(\mathrm{d}\varphi-\omega_H\mathrm{d}t)^2\right], \tag{A9}$$

where (x^A) stands for the coordinates spanning $\mathcal{H}$: $(x^A)=(t,\theta,\varphi)$. This metric is clearly degenerate, with the degeneracy direction along $\ell^A=(1,0,\omega_H)$. We thus recover the fact that $\mathcal{H}$ is a null hypersurface.

Setting $\mathrm{d}t=0$ in the line element (A9), we get the induced metric $\boldsymbol{q}$ in the 2-surfaces $\mathcal{S}_t$ that foliate $\mathcal{H}$:

$$q_{ab}\mathrm{d}x^a\mathrm{d}x^b = 2mr_H\left[(1-a\omega_H\sin^2\theta)\mathrm{d}\theta^2 + \frac{\sin^2\theta}{1-a\omega_H\sin^2\theta}\mathrm{d}\varphi^2\right], \tag{A10}$$

where (x^a) stands for the coordinates spanning $\mathcal{S}_t$: $(x^a)=(\theta,\varphi)$. The metric $\boldsymbol{q}$ is clearly positive definite; hence, the 2-surfaces are spacelike. From (A10), we read immediately the determinant of $\boldsymbol{q}$ with respect to the coordinates (θ,φ):

$$\sqrt{q} = 2mr_H\sin\theta. \tag{A11}$$

APPENDIX B: FLUX INTEGRALS ON A HYPERSURFACE

Let Σ be an oriented hypersurface in the spacetime $(\mathcal{M},\boldsymbol{g})$. From the very definition of the integral of a 3-form over a three-dimensional manifold, we have

$$\begin{aligned}\int_\Sigma \boldsymbol{\epsilon}(\vec{\boldsymbol{P}}) &= \int_\Sigma \boldsymbol{\epsilon}(\vec{\boldsymbol{P}})(\mathrm{d}\vec{\boldsymbol{x}}_{(1)},\mathrm{d}\vec{\boldsymbol{x}}_{(2)},\mathrm{d}\vec{\boldsymbol{x}}_{(3)})\\ &= \int_\Sigma \boldsymbol{\epsilon}(\vec{\boldsymbol{P}},\mathrm{d}\vec{\boldsymbol{x}}_{(1)},\mathrm{d}\vec{\boldsymbol{x}}_{(2)},\mathrm{d}\vec{\boldsymbol{x}}_{(3)}),\end{aligned} \tag{B1}$$

where the last equality follows from the definition (4.4) of $\boldsymbol{\epsilon}(\vec{\boldsymbol{P}})$ and $(\mathrm{d}\vec{\boldsymbol{x}}_{(1)},\mathrm{d}\vec{\boldsymbol{x}}_{(2)},\mathrm{d}\vec{\boldsymbol{x}}_{(3)})$ are infinitesimal vectors forming an elementary right-handed parallelepiped on Σ.

1. Case of a spacelike or timelike hypersurface

If Σ is spacelike or timelike, we may introduce the unit normal $\vec{\boldsymbol{m}}$ that is compatible with Σ's orientation [i.e., such that the orientation is given by the 3-form $\boldsymbol{\epsilon}(\vec{\boldsymbol{m}})=\boldsymbol{\epsilon}(\vec{\boldsymbol{m}},.,.,.)$; cf. Sec. IVA]. The orthogonal decomposition of $\vec{\boldsymbol{P}}$ with respect to Σ is then

$$\vec{\boldsymbol{P}} = \pm(P_\mu m^\mu)\vec{\boldsymbol{m}} + \vec{\boldsymbol{P}}_\parallel, \tag{B2}$$

where $\pm$ is $+$ $(-)$ if Σ is timelike (spacelike) and $\vec{\boldsymbol{P}}_\parallel$ is tangent to Σ. The four vectors $\vec{\boldsymbol{P}}_\parallel$, $\mathrm{d}\vec{\boldsymbol{x}}_{(1)}$, $\mathrm{d}\vec{\boldsymbol{x}}_{(2)}$, and $\mathrm{d}\vec{\boldsymbol{x}}_{(3)}$ cannot be linearly independent, being all tangent to Σ, so that $\boldsymbol{\epsilon}(\vec{\boldsymbol{P}}_\parallel,\mathrm{d}\vec{\boldsymbol{x}}_{(1)},\mathrm{d}\vec{\boldsymbol{x}}_{(2)},\mathrm{d}\vec{\boldsymbol{x}}_{(3)})=0$. Hence,

$$\boldsymbol{\epsilon}(\vec{\boldsymbol{P}},\mathrm{d}\vec{\boldsymbol{x}}_{(1)},\mathrm{d}\vec{\boldsymbol{x}}_{(2)},\mathrm{d}\vec{\boldsymbol{x}}_{(3)}) = \pm(P_\mu m^\mu)\boldsymbol{\epsilon}(\vec{\boldsymbol{m}},\mathrm{d}\vec{\boldsymbol{x}}_{(1)},\mathrm{d}\vec{\boldsymbol{x}}_{(2)},\mathrm{d}\vec{\boldsymbol{x}}_{(3)}). \tag{B3}$$

Now, since $\vec{\boldsymbol{m}}$ is a unit vector,

$$\mathrm{d}V := \boldsymbol{\epsilon}(\vec{\boldsymbol{m}},\mathrm{d}\vec{\boldsymbol{x}}_{(1)},\mathrm{d}\vec{\boldsymbol{x}}_{(2)},\mathrm{d}\vec{\boldsymbol{x}}_{(3)}) \tag{B4}$$

is nothing but the volume of the elementary parallelepiped formed by $(\mathrm{d}\vec{\boldsymbol{x}}_{(1)},\mathrm{d}\vec{\boldsymbol{x}}_{(2)},\mathrm{d}\vec{\boldsymbol{x}}_{(3)})$ with respect to the 3-metric $\boldsymbol{\gamma}$ induced by $\boldsymbol{g}$ on Σ [for Σ_{ext}, $\boldsymbol{\gamma}$ is denoted by $\boldsymbol{h}$ in (4.11)]. By combining (B1), (B3), and (B4), we get

$$\int_\Sigma \boldsymbol{\epsilon}(\vec{\boldsymbol{P}}) = \pm\int_\Sigma P_\mu m^\mu \mathrm{d}V. \tag{B5}$$

This establishes the second equalities in (4.8), (4.9), and (4.11).

Let (x^1,x^2,x^3) be a coordinate system on Σ and let us choose the $\mathrm{d}\vec{\boldsymbol{x}}_{(i)}$'s as the corresponding elementary displacements:

$$\mathrm{d}\vec{\boldsymbol{x}}_{(1)} = \mathrm{d}x^1\frac{\partial}{\partial x^1},\qquad \mathrm{d}\vec{\boldsymbol{x}}_{(2)} = \mathrm{d}x^2\frac{\partial}{\partial x^2},$$
$$\mathrm{d}\vec{\boldsymbol{x}}_{(3)} = \mathrm{d}x^3\frac{\partial}{\partial x^3}.$$

Then

$$\mathrm{d}V = \sqrt{|\gamma|}\mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3, \tag{B6}$$

where $\gamma=\det(\gamma_{ij})$, the γ_{ij}'s being the components of the induced 3-metric on Σ. This established the third equalities in (4.8), (4.9), and (4.11).

2. Case of null hypersurface

Here we consider that $\Sigma=\Delta\mathcal{H}$, but the results are valid for any null hypersurface. Since $\Delta\mathcal{H}$ is null, there is no orthogonal decomposition of $\vec{\boldsymbol{P}}$ of the type (B2). Let us consider instead the slicing of $\Delta\mathcal{H}$ by the 2-spheres $\mathcal{S}_t$ of constant t (cf. Sec. III B). Then we have the following unique decomposition of $\vec{\boldsymbol{P}}$:

$$\vec{\boldsymbol{P}} = -(P_\mu\ell^\mu)\vec{\boldsymbol{k}} - (P_\mu k^\mu)\vec{\boldsymbol{\ell}} + \vec{\boldsymbol{P}}_\parallel, \tag{B7}$$

where $\vec{P}_{\parallel}$ is tangent to $\mathcal{S}_t$. This decomposition follows from the fact that $\vec{k}$ and $\vec{\ell}$ generate the 2-plane orthogonal to $\mathcal{S}_t$ and from the normalization relation (3.4).

Let us choose the elementary parallelepiped $(\mathrm{d}\vec{x}_{(1)}, \mathrm{d}\vec{x}_{(2)}, \mathrm{d}\vec{x}_{(3)})$ on $\Delta\mathcal{H}$ such that

$$\mathrm{d}\vec{x}_{(1)} = \mathrm{d}t\vec{\ell}$$

and $\mathrm{d}\vec{x}_{(2)}$ and $\mathrm{d}\vec{x}_{(3)}$ are tangent to $\mathcal{S}_t$. The integrand in (B1) is then

$$\boldsymbol{\epsilon}(\vec{P}, \mathrm{d}\vec{x}_{(1)}, \mathrm{d}\vec{x}_{(2)}, \mathrm{d}\vec{x}_{(3)}) = \mathrm{d}t\boldsymbol{\epsilon}(\vec{P}, \vec{\ell}, \mathrm{d}\vec{x}_{(2)}, \mathrm{d}\vec{x}_{(3)}). \tag{B8}$$

Now, from (B7),

$$\begin{aligned}\boldsymbol{\epsilon}(\vec{P}, \vec{\ell}, \mathrm{d}\vec{x}_{(2)}, \mathrm{d}\vec{x}_{(3)}) &= -(P_\mu \ell^\mu)\boldsymbol{\epsilon}(\vec{k}, \vec{\ell}, \mathrm{d}\vec{x}_{(2)}, \mathrm{d}\vec{x}_{(3)}) \\ &\quad - (P_\mu k^\mu)\underbrace{\boldsymbol{\epsilon}(\vec{\ell}, \vec{\ell}, \mathrm{d}\vec{x}_{(2)}, \mathrm{d}\vec{x}_{(3)})}_{0} \\ &\quad + \underbrace{\boldsymbol{\epsilon}(\vec{P}_{\parallel}, \vec{\ell}, \mathrm{d}\vec{x}_{(2)}, \mathrm{d}\vec{x}_{(3)})}_{0} \\ &= -(P_\mu \ell^\mu)\boldsymbol{\epsilon}(\vec{k}, \vec{\ell}, \mathrm{d}\vec{x}_{(2)}, \mathrm{d}\vec{x}_{(3)}).\end{aligned}$$

Therefore, we may rewrite (B1) as

$$\int_{\Delta\mathcal{H}} \boldsymbol{\epsilon}(\vec{P}) = -\int_{\Delta\mathcal{H}} P_\mu \ell^\mu \mathrm{d}V, \tag{B9}$$

with

$$\mathrm{d}V = \boldsymbol{\epsilon}(\vec{k}, \vec{\ell}, \mathrm{d}\vec{x}_{(2)}, \mathrm{d}\vec{x}_{(3)})\mathrm{d}t. \tag{B10}$$

This establishes the second equality in (4.10).

Let (y^1, y^2) be a coordinate system on $\mathcal{S}_t$ and let us choose the $\mathrm{d}\vec{x}_{(2)}$ and $\mathrm{d}\vec{x}_{(3)}$ as the corresponding elementary displacements:

$$\mathrm{d}\vec{x}_{(2)} = \mathrm{d}y^1 \frac{\partial}{\partial y^1}, \qquad \mathrm{d}\vec{x}_{(3)} = \mathrm{d}y^2 \frac{\partial}{\partial y^2}.$$

We have then

$$\begin{aligned}\mathrm{d}V &= \boldsymbol{\epsilon}(\vec{k}, \vec{\ell}, \mathrm{d}\vec{x}_{(2)}, \mathrm{d}\vec{x}_{(3)})\mathrm{d}t \\ &= \boldsymbol{\epsilon}(\vec{k}, \vec{\ell}, \partial/\partial y_1, \partial/\partial y_2)\mathrm{d}t\mathrm{d}y^1\mathrm{d}y^2 \\ &= \sqrt{-\tilde{g}}\mathrm{d}t\mathrm{d}y^1\mathrm{d}y^2,\end{aligned} \tag{B11}$$

where $\tilde{g}$ is the determinant of the components of $\boldsymbol{g}$ in the basis $(\vec{k}, \vec{\ell}, \partial/\partial y_1, \partial/\partial y_2)$. Given the definitions of $\vec{k}$ and $\boldsymbol{q}$, these components are

$$\tilde{g}_{\alpha\beta} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & q_{11} & q_{12} \\ 0 & 0 & q_{12} & q_{22} \end{pmatrix}. \tag{B12}$$

Hence $\tilde{g} = -q$, with $q \coloneqq \det(q_{ab})$, and (B11) becomes

$$\mathrm{d}V = \sqrt{q}\mathrm{d}t\mathrm{d}y^1\mathrm{d}y^2. \tag{B13}$$

This establishes the third equality in (4.10).

APPENDIX C: CALCULATION OF PARTICLE ENERGY AS A FLUX THROUGH SOME HYPERSURFACE

1. Case of a spacelike hypersurface

As shown in Sec. VI A, the particle energy at the event A_1 on Σ_1 is

$$\begin{aligned}E_1 = \mathfrak{m}_1 \int_{\Sigma_1} \int_{-\infty}^{+\infty} &\delta_{A(\tau)}(M) g_\mu{}^\rho(M, A(\tau))(u_1)_\rho(\tau) g_\nu{}^\sigma \\ &\times (M, A(\tau))(u_1)_\sigma(\tau)\eta^\mu(M) n_1^\nu(M)\sqrt{\gamma}\mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3\mathrm{d}\tau.\end{aligned} \tag{C1}$$

Thanks to the Dirac distribution, only the terms for which $M = A(\tau)$ contribute to the above integral. We may then drop the parallel propagators and write

$$\begin{aligned}E_1 = \mathfrak{m}_1 \int_{\Sigma_1} \int_{-\infty}^{+\infty} &\delta_{A(\tau)}(M)(u_1)_\mu(\tau)\eta^\mu(M)(u_1)_\nu(\tau) n_1^\nu \\ &\times (M)\sqrt{\gamma}\mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3\mathrm{d}\tau.\end{aligned}$$

Let us introduce in the vicinity of A_1 a coordinate system (t, x^1, x^2, x^3) such that Σ_1 is the hypersurface $t = 0$ and t increases towards the future. Then the normal $\vec{n}_1$ is collinear to the gradient of t: $(n_1)_\alpha = -N\nabla_\alpha t$, the coefficient $N > 0$ being called the *lapse function*. We have then $(n_1)_\alpha = (-N, 0, 0, 0)$ and

$$(u_1)_\nu n_1^\nu = (n_1)_\nu u_1^\nu = -N u_1^0.$$

Hence,

$$\begin{aligned}E_1 = -\mathfrak{m}_1 \int_{\Sigma_1} \int_{-\infty}^{+\infty} &\delta_{A(\tau)}(M)\eta_\mu(M) u_1^\mu(\tau) \\ &\times N\sqrt{\gamma}\mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3 u_1^0\mathrm{d}\tau.\end{aligned}$$

Since the particle's worldline is timelike and therefore never tangent to Σ_1, we may use t as a regular parameter along it and perform the change of variable $\tau \to t$ in the above integral. Taking into account that $u_1^0 = \mathrm{d}t/\mathrm{d}\tau$ (from the very definition of a 4-velocity), we get

$$E_1 = -\mathfrak{m}_1 \int_{\Sigma_1} \int_{-\infty}^{+\infty} \delta_{A(t)}(M) \eta_\mu(M) u_1^\mu(t) \times N\sqrt{\gamma} \mathrm{d}x^1 \mathrm{d}x^2 \mathrm{d}x^3 \mathrm{d}t. \tag{C2}$$

Within the coordinate system (t, x^1, x^2, x^3), the coordinates of $A(t)$ are $(t, z^1(t), z^2(t), z^3(t))$ and those of M are $(0, x^1, x^2, x^3)$ (for $M \in \Sigma_1$). Therefore, using (6.3) along with the identity $\sqrt{-g} = N\sqrt{\gamma}$ [see, e.g., Eq. (5.55) in [33]], we obtain

$$E_1 = -m_1 \int_{\Sigma_1} \int_{-\infty}^{+\infty} \delta(-t)\delta(x^1 - z^1(t))\delta(x^2 - z^2(t)) \times \delta(x^3 - z^3(t)) \eta_\mu(M) u_1^\mu(t) \mathrm{d}x^1 \mathrm{d}x^2 \mathrm{d}x^3 \mathrm{d}t.$$

Since $\delta(-t) = \delta(t)$, performing the integration on t leads to

$$\begin{aligned} E_1 &= -\mathfrak{m}_1 \int_{\Sigma_1} \delta(x^1 - z^1(0))\delta(x^2 - z^2(0))\delta(x^3 - z^3(0)) \eta_\mu(M) u_1^\mu(0) \mathrm{d}x^1 \mathrm{d}x^2 \mathrm{d}x^3 \\ &= -\mathfrak{m}_1 \eta_\mu(0, z^1(0), z^2(0), z^3(0)) u_1^\mu(0). \end{aligned}$$

Since $(0, z^1(0), z^2(0), z^3(0))$ are the coordinates of A_1 and $u_1^\mu(0)$ are the components of $\vec{u}_1$ at A_1, we conclude that

$$E_1 = -\mathfrak{m}_1 (\eta_\mu u_1^\mu)|_{A_1} = -\mathfrak{m}_1 \eta_\mu u_1^\mu. \tag{C3}$$

2. Case of a null hypersurface

In Sec. VI A we obtained for the energy of the particle crossing the event horizon

$$\Delta E_H = \mathfrak{m}_* \int_{\Delta\mathcal{H}} \int_{-\infty}^{\infty} \delta_{A(\tau)}(M)(u_*)_\mu(\tau)\eta^\mu(M)(u_*)_\nu(\tau)\ell^\nu \times (M)\sqrt{q} \mathrm{d}t \mathrm{d}y^1 \mathrm{d}y^2 \mathrm{d}\tau. \tag{C4}$$

Note that, for the same reasons as above, we have dropped the parallel propagators. Let us introduce in the vicinity of A_H a coordinate system (w, t, y^1, y^2) such that $\mathcal{H}$ is the hypersurface $w = 0$, $\vec{k} = \partial/\partial w$ on $\mathcal{H}$ and $\vec{\ell} = \partial/\partial t$ on $\mathcal{H}$. Let us expand $\vec{u}_*$ in the associated coordinate basis:

$$\vec{u}_* = u_*^0 \vec{k} + u_*^1 \vec{\ell} + u_*^2 \frac{\partial}{\partial y^1} + u_*^3 \frac{\partial}{\partial y^2}.$$

We have then, given (3.4) and the orthogonality of $\vec{\ell}$ to itself and to $\partial/\partial y^1$ and $\partial/\partial y^2$,

$$(u_*)_\nu \ell^\nu = u_*^\nu \ell_\nu = -u_*^0 = -\frac{\mathrm{d}w}{\mathrm{d}\tau}. \tag{C5}$$

Since the worldline of $\mathcal{P}_*$ is crossing $\mathcal{H}$, we may use w as a regular parameter on it and perform the change of variable $\tau \to w$ in the integral (C4), taking advantage of (C5). Therefore

$$\Delta E_H = -\mathfrak{m}_* \int_{\Delta\mathcal{H}} \int_{-\infty}^{+\infty} \delta_{A(w)}(M) \eta_\mu(M) u_*^\mu(w) \sqrt{q} \mathrm{d}t \mathrm{d}y^1 \mathrm{d}y^2 \mathrm{d}w.$$

Within the coordinate system (w, t, y^1, y^2), the coordinates of $A(w)$ are $(w, z^1(w), z^2(w), z^3(w))$ and those of M are $(0, t, y^1, y^2)$ (for $M \in \Delta\mathcal{H}$). Therefore, using (6.3), we obtain

$$\Delta E_H = -\mathfrak{m}_* \int_{\Delta\mathcal{H}} \int_{-\infty}^{+\infty} \delta(-w)\delta(t - z^1(w))\delta(y^1 - z^2(w)) \times \delta(y^2 - z^3(w)) \eta_\mu(M) u_*^\mu(w) \frac{\sqrt{q}}{\sqrt{-g}} \mathrm{d}t \mathrm{d}y^1 \mathrm{d}y^2 \mathrm{d}w.$$

Performing the integration on w, we get

$$\Delta E_H = -\mathfrak{m}_* \int_{\Delta\mathcal{H}} \delta(t - z^1(0))\delta(y^1 - z^2(0))\delta(y^2 - z^3(0)) \times \eta_\mu(M) u_*^\mu(0) \frac{\sqrt{q}}{\sqrt{-g}} \mathrm{d}t \mathrm{d}y^1 \mathrm{d}y^2.$$

On $\Delta\mathcal{H}$, the components of the metric tensor with respect to the coordinates (w, t, y^1, y^2) are given by (B12), from which we deduce that $\sqrt{-g} = \sqrt{q}$. Noticing that $(0, z^1(0), z^2(0), z^3(0))$ are the coordinates of A_H, we conclude that

$$\Delta E_H = -\mathfrak{m}_* (\eta_\mu u_*^\mu)|_{A_H} = -\mathfrak{m}_* \eta_\mu u_*^\mu. \tag{C6}$$

APPENDIX D: ENERGY AND ANGULAR-MOMENTUM CONSERVATION LAWS IN ADAPTED COORDINATES

In this appendix, we derive the energy conservation law (4.12), as well as the angular-momentum one (4.19), by a direct calculation within *adapted* coordinates $(x^\alpha) = (t, r, \theta, \varphi)$, as defined in Sec. IV C. The starting point is the covariant energy-momentum conservation law $\nabla_\mu T^\mu{}_\alpha = 0$, which can be expressed in terms of partial derivatives thanks to a standard formula for the covariant divergence of a symmetric tensor tensor field:

$$\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\mu} (\sqrt{-g} T^\mu{}_\alpha) - \frac{1}{2} \frac{\partial g_{\mu\nu}}{\partial x^\alpha} T^{\mu\nu} = 0. \tag{D1}$$

For $\alpha = 0$ and $\alpha = 3$, the second term in the left-hand side vanishes, due to the spacetime symmetries ($\partial g_{\mu\nu}/\partial t = 0$ and $\partial g_{\mu\nu}/\partial \varphi = 0$). We are thus left with

$$\frac{\partial}{\partial x^\mu} (\sqrt{-g} T^\mu{}_\alpha) = 0 \quad (\alpha = 0, 3). \tag{D2}$$

Let us integrate this equation over the coordinate 4-volume formed by the Cartesian product $[t_1, t_2] \times [r_H, r_{\text{ext}}] \times [0, \pi] \times [0, 2\pi]$. This corresponds to the coordinate ranges of the spacetime 4-volume enclosed in the hypersurface $\mathcal{V} \coloneqq \Sigma_1 \cup \Delta\mathcal{H} \cup \Sigma_2 \cup \Sigma_{\text{ext}}$ considered in Sec. IV and to which the coordinates (t, r, θ, φ) are adapted. We get

$$\int_{t=t_1}^{t=t_2} \int_{r=r_H}^{r=r_{\text{ext}}} \int_{\theta=0}^{\theta=\pi} \int_{\varphi=0}^{\varphi=2\pi} \frac{\partial}{\partial x^\mu} (\sqrt{-g} T^\mu{}_\alpha) \times \mathrm{d}t\mathrm{d}r\mathrm{d}\theta\mathrm{d}\varphi = 0 (\alpha = 0, 3). \tag{D3}$$

Since the integral bounds are independent from one another, we may permute the various integrals and use the identities

$$\int_{t=t_1}^{t=t_2} \frac{\partial}{\partial t} (\sqrt{-g} T^t{}_\alpha) \mathrm{d}t = (\sqrt{-g} T^t{}_\alpha)_{t=t_2} - (\sqrt{-g} T^t{}_\alpha)_{t=t_1} \tag{D4}$$

$$\int_{r=r_H}^{r=r_{\text{ext}}} \frac{\partial}{\partial r} (\sqrt{-g} T^r{}_\alpha) \mathrm{d}r = (\sqrt{-g} T^r{}_\alpha)_{r=r_{\text{ext}}} - (\sqrt{-g} T^r{}_\alpha)_{r=r_H} \tag{D5}$$

$$\int_{\theta=0}^{\theta=\pi} \frac{\partial}{\partial \theta} (\sqrt{-g} T^\theta{}_\alpha) \mathrm{d}\theta = (\sqrt{-g} T^\theta{}_\alpha)_{\theta=\pi} - (\sqrt{-g} T^\theta{}_\alpha)_{\theta=0} = 0 \tag{D6}$$

$$\int_{\varphi=0}^{\varphi=2\pi} \frac{\partial}{\partial \varphi} (\sqrt{-g} T^\varphi{}_\alpha) \mathrm{d}\varphi = (\sqrt{-g} T^\varphi{}_\alpha)_{\varphi=2\pi} - (\sqrt{-g} T^\varphi{}_\alpha)_{\varphi=0} = 0. \tag{D7}$$

The "$= 0$" in (D6) results from $\sqrt{-g} = 0$ at $\theta = 0$ and $\theta = \pi$, as a consequence of regularity properties of spherical coordinates, while the "$= 0$" of (D7) results from the 2π-periodicity associated with the coordinate φ. Taking into account ((D4)–(D7), Eq. (D3) becomes

$$\int_{\Sigma_2} T^t{}_\alpha \sqrt{-g} \mathrm{d}r\mathrm{d}\theta\mathrm{d}\varphi - \int_{\Sigma_1} T^t{}_\alpha \sqrt{-g} \mathrm{d}r\mathrm{d}\theta\mathrm{d}\varphi + \int_{\Sigma_{\text{ext}}} T^r{}_\alpha \sqrt{-g} \mathrm{d}t\mathrm{d}\theta\mathrm{d}\varphi - \int_{\Delta\mathcal{H}} T^r{}_\alpha \sqrt{-g} \mathrm{d}t\mathrm{d}\theta\mathrm{d}\varphi = 0 \quad (\alpha = 0, 3). \tag{D8}$$

For $\alpha = 0$, we recognize the energy conservation law (4.12), the four integrals being, respectively, $-E_2$, E_1, $-\Delta E_{\text{ext}}$, and $-\Delta E_H$ as given by (4.20)–(4.26). For $\alpha = 3$, we get the angular-momentum conservation law (4.19), the four integrals being, respectively, J_2, $-J_1$, ΔJ_{ext}, and ΔJ_H as given by (4.27)–(4.29).

Note that in the above derivation, as in the geometrical derivation of Sec. IV, we have not assumed that the energy-momentum tensor $\boldsymbol{T}$ obeys the spacetime symmetries.

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