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The use of the quaternions for describing the Earth's rotation

C. Bizouard¹, Y. Cheng¹

Abstract

We derive the quaternions expressing the rotation transformation between celestial and terrestrial reference systems as functions of the parameters giving the position of the Celestial Intermediate Pole (CIP) in those two systems, and of the rotation angle. A first version is associated with the common Earth Orientation Parameters (EOP) recommended by the International Earth Rotation and Reference System Service (IERS) and IAU; a second version uses the direction cosines of the CIP in the International Terrestrial Reference System (ITRS) instead of the conventional pole coordinates. Whereas matrix and quaternion methods are numerically comparable, the quaternion formulation offers a very elegant and concise analytical representation, that does not exist for Earth rotation matrix and that can be easily programmed. In our view, this quaternion representation reinforces the scope of the nowadays parameters adopted for describing Earth rotation.

Keywords Earth rotation · Quaternion · Coordinate transformation

1 Introduction

Widely used to control the attitude of a spacecraft from the gyroscopic measurements of its angular velocity vector, the quaternions have been very little applied in the Earth rotation studies. In his doctoral thesis, Bizouard (1996) developed alternative Earth rotation parameters based upon quaternions. The paper of Capitaine et al. (2003) briefly discusses the possibility to formulate the precession by quaternions. In 2006 Svehla (2018) showed how quaternions facilitate the transition of the Earth rotation transformations between two successive close epochs as long as the rotation vector is known. But, to our knowledge, the rotation transformation between the International Terrestrial Reference System (ITRS) and the Geocentric Celestial Reference System (GCRS) has never been fully expressed in terms of quaternion.

That rotation transformation is commonly described through a sequence of elementary rotations around different axes. Until the 2000, the conventional sequence was

based upon the astronomical planes and direction involved in the dynamical description of the precession–nutation: the ecliptic plane, the mean and true equatorial planes with the descending nodes with the ecliptic, called mean and true equinoctial points respectively. In the modern representation, adopted by IAU in 2000, the ecliptic plane and subsequent equinoctial points were given up for the benefit of a single intermediate plane, namely the true equatorial plane, perpendicular to the Celestial Intermediate Pole (CIP) and two intermediate origins or non-rotating origins on this plane allowing to clearly define UT1.

In that new representation, developed and advocated in the 1980s and 1990s by Capitaine et al. (1986), the Celestial Intermediate Pole (called Celestial Ephemeris Pole until 2000) plays a central role. As in IERS conventions (Petit and Luzum 2010), the matrix M transforming terrestrial coordinates into celestial coordinates is structured around the CIP according to the compact expression

$$M(t) = Q(t)R(t)W(t), \quad (1)$$

where

- $W(t) = R_3(-s')R_2(x)R_1(y)$ is the polar motion matrix depending on (i) the elementary rotation angles (x, y) , abusively called "pole coordinates" of the CIP in the ITRS—see Sect. 4 for the relation between (x, y) and the true pole coordinates (x_p, y_p) (ii) the locator of Ter-

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restrial Intermediate Origin (TIO) s' , transforming ITRS to Terrestrial Intermediate Reference System (TIRS)

- $R(t) = R_3(-\text{ERA}) = R_3(-\theta)$ accounts for the rotation around the CIP between Terrestrial Intermediate Origin (TIO) and Celestial Intermediate Origin (CIO) given by $\theta = \Omega \text{UT1} + \theta_0$ where Ω is the reference Earth rotation angular velocity, UT1 is the Earth rotation time, and θ_0 a given constant angle, transforming Terrestrial Intermediate Reference System to Celestial Intermediate Reference System (CIRS)
- $Q(t) = PN(X, Y)R_3(s)$ where

$$PN(X, Y) = \begin{bmatrix} 1 - aX^2 & -aXY & X \\ -aXY & 1 - aY^2 & Y \\ -X & -Y & 1 - a(X^2 + Y^2) \end{bmatrix} \quad (2)$$

gives the coupled precession–nutation matrix as a function of the direction cosines X and Y of the CIP and the CIO locator s in the GCRS. Capitaine (1990) put forward the simplicity of this representation with respect to the angular description, and developed the X and Y coordinates as functions of time.

Astrogeodetic observations allows to determine the irregular parts of parameters mentioned above: the pole coordinates x, y , the non-uniform part of θ , that is $\Omega(\text{UT1} - \text{UTC})$, and corrections (dX, dY) to the modeled part $(X_{\text{mod}}, Y_{\text{mod}})$ of (X, Y) coordinates, called celestial pole offsets. Those five parameters are the so-called Earth Orientation Parameters (EOP).

In IERS Conventions (2010), Chapter 5, IERS recommends the expression of the precession–nutation matrix based upon the celestial coordinates (X, Y) of the CIP (Petit and Luzum 2010). In contrast to angular description of the precession–nutation, the analytical representation with X, Y parameters does not derive directly from the precession–nutation theory where the introduction of the ecliptic plane is unavoidable, and it contains more terms. That is why this description faced many oral criticisms during international meetings devoted to Earth’s rotation in the beginning of the 2000’s and is still not adopted universally in the computer codes for handling space geodetic data. However, as this paper will show, despite the lack of dynamical meaning of the direction cosines X and Y of the CIP in the GCRS, they provide a very elegant and concise analytical representation in terms of quaternions, that does not exist in the matrix formulation.

First, according to the properties of rotation quaternions established in Appendix B, we derive the analytical expression of the Earth rotation quaternion (Sect. 2), validate it with numerical computation (Sect. 3). Finally, in Sect. 4, we gen-

eralize this expression to the direction cosines of the CIP in the ITRS instead of the conventional pole coordinates.

A reader not familiar with quaternions is advised to read first Appendices A (definition and properties of the quaternions) and B (description of a rotation by a unit quaternion and corresponding transformation coordinates) or any textbook/internet resources related to those mathematical objects, for instance de Casteljalou (1997).

2 Earth rotation quaternion

Let [GCRS] be the coordinates of a given vector in the GCRS, and [ITRS] its corresponding coordinates in the ITRS. The coordinate transformation from the ITRS to the GCRS reads [GCRS] = M [ITRS] where M is expressed by the matrix (Petit and Luzum 2010)

$$M = R_3(-E)R_2(-d)R_3(E)R_3(s)R_3(-\theta)R_3(-s')R_2(x)R_1(y), \quad (3)$$

where d, E are the colatitude and longitude of the Celestial Intermediate Pole in the GCRS.

According to Appendix B.4, the coordinate transformation of the precession–nutation sequence, $R_3(-E)R_2(-d)R_3(E)$, is given by the quaternion

$$q_{PN} = q_3(E) \otimes q_2(-d) \otimes q_3(-E) = \begin{bmatrix} \cos \frac{E}{2} & 0 & 0 & \sin \frac{E}{2} \\ 0 & \cos \frac{d}{2} & -\sin \frac{d}{2} & 0 \\ 0 & \sin \frac{d}{2} & \cos \frac{d}{2} & 0 \\ \sin \frac{E}{2} & 0 & 0 & \cos \frac{E}{2} \end{bmatrix} = \begin{bmatrix} \cos \frac{d}{2} \cos \frac{E}{2} \\ \sin \frac{d}{2} \cos \frac{E}{2} \\ -\sin \frac{d}{2} \sin \frac{E}{2} \\ \cos \frac{d}{2} \sin \frac{E}{2} \end{bmatrix} \quad (4)$$

According to the quaternion form (B7), that expression means that the precession–nutation transformation is a rotation of angle $-d/2$ around the axis of director cosine $(-\sin E = \cos(E + \pi/2), \cos E = \sin(E + \pi/2), 0)$, thus lying in the XY plane at an angle $E + \pi/2$ to OX . It can be expressed in function of the Cartesian coordinates of the CIP in the ICRF, according to the relations given by the IERS Conventions (Petit and Luzum 2010):

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} \sin d \cos E \\ \sin d \sin E \\ \cos d \end{bmatrix}, \quad (5)$$

we have

$$2 \cos^2 \frac{d}{2} - 1 = Z \quad 2 \sin^2 \frac{d}{2} - 1 = -Z$$

$$\cos \frac{d}{2} = \sqrt{\frac{1+Z}{2}} \quad \text{because}$$

$$\begin{aligned}
0 &\leq \frac{d}{2} \leq \frac{\pi}{2} \\
\sin \frac{d}{2} &= \sqrt{\frac{1-Z}{2}} \quad \text{because} \\
0 &\leq \frac{d}{2} \leq \frac{\pi}{2} \\
\sin \frac{d}{2} \cos E &= \sin \frac{d}{2} \frac{X}{\sin d} \\
&= \frac{X}{2 \cos \frac{d}{2}} = \frac{X}{\sqrt{2(1+Z)}} \\
\sin \frac{d}{2} \sin E &= \sin \frac{d}{2} \frac{Y}{\sin d} \\
&= \frac{Y}{2 \cos \frac{d}{2}} = \frac{Y}{\sqrt{2(1+Z)}}, \tag{6}
\end{aligned}$$

and (4) has the shortest expression

$$q_{PN} = \frac{1}{\sqrt{2(1+Z)}} \begin{pmatrix} 1+Z \\ Y \\ -X \\ 0 \end{pmatrix}. \tag{7}$$

This simple expression makes the direction cosines of the CIP, as introduced in Capitaine et al. (1986, 1990, 2000), quite remarkable for describing the precession–nutation independently from the ecliptic plane.

The sequence $R_2(x)R_1(y)$ is the rotation transformation associated with the polar wobble, and given by the quaternion

$$\begin{aligned}
q_{W_1} &= q_1(y) \otimes q_2(x) \\
&= \begin{pmatrix} \cos \frac{y}{2} & \cos \frac{x}{2} \\ \sin \frac{y}{2} & 0 \\ 0 & \sin \frac{x}{2} \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} \cos \frac{x}{2} \cos \frac{y}{2} \\ \cos \frac{x}{2} \sin \frac{y}{2} \\ \sin \frac{x}{2} \cos \frac{y}{2} \\ \sin \frac{x}{2} \sin \frac{y}{2} \end{pmatrix}. \tag{8}
\end{aligned}$$

Neglecting second-order terms with x^2 , xy , y^2 and terms of higher orders (smaller than $(10^{-6})^2 = 10^{-12}$ rad = 0.2 μ as), we obtain

$$\tilde{q}_{W_1} = \begin{pmatrix} 1 \\ \frac{y}{2} \\ \frac{x}{2} \\ 0 \end{pmatrix}. \tag{9}$$

Only within this approximation, $q_1(y)$ and $q_2(x)$ becomes commutative. The non-commutativity of quaternion $q_1(y)$ and $q_2(x)$ is associated with the sign of the last quaternion component in (8), here $c_w = \sin \frac{x}{2} \sin \frac{y}{2}$. For the inverse product $q_2(x) \otimes q_1(y)$, c_w gets the opposite sign.

The diurnal rotation $R_3(-s' - \theta + s)$ including the slow motion of the TIO and CIO in the intermediate equatorial plane is described by the quaternion

$$\begin{aligned}
q_{R_1} &= q_3(s - \theta - s') \\
&= \begin{pmatrix} \cos \frac{\theta'}{2} \\ 0 \\ 0 \\ -\sin \frac{\theta'}{2} \end{pmatrix}, \tag{10}
\end{aligned}$$

with $\theta' = -(s - \theta - s') = \theta + s' - s$.

So, to sum up the total transformation (3) is expressed by the quaternion

$$\begin{aligned}
\tilde{q}_M &= \tilde{q}_{W_1} \otimes q_{R_1} \otimes q_{PN} \\
&= \begin{pmatrix} 1 \\ \frac{y}{2} \\ \frac{x}{2} \\ 0 \end{pmatrix} \begin{pmatrix} \cos \frac{\theta'}{2} \\ 0 \\ 0 \\ -\sin \frac{\theta'}{2} \end{pmatrix} \frac{1}{\sqrt{2(1+Z)}} \begin{pmatrix} 1+Z \\ Y \\ -X \\ 0 \end{pmatrix}, \tag{11}
\end{aligned}$$

that is

$$\tilde{q}_M = \frac{1}{\sqrt{2(1+Z)}} \begin{pmatrix} \cos \frac{\theta'}{2} (X \frac{x}{2} - Y \frac{y}{2} + 1 + Z) + \sin \frac{\theta'}{2} (X \frac{y}{2} + Y \frac{x}{2}) \\ \cos \frac{\theta'}{2} (Y + (1+Z) \frac{y}{2}) + \sin \frac{\theta'}{2} (-X - (1+Z) \frac{x}{2}) \\ \cos \frac{\theta'}{2} (-X + (1+Z) \frac{x}{2}) + \sin \frac{\theta'}{2} (-Y + (1+Z) \frac{y}{2}) \\ \cos \frac{\theta'}{2} (-X \frac{y}{2} + Y \frac{x}{2}) + \sin \frac{\theta'}{2} (X \frac{x}{2} - Y \frac{y}{2} - (1+Z)) \end{pmatrix}. \tag{12}$$

If we take the non-approximate expression (8) for the polar motion quaternion q_W , a quite simple analytical derivation is still possible, yielding the fully exact quaternion

$$q_M = \frac{1}{\sqrt{2(1+Z)}} \begin{pmatrix} \cos \frac{\theta'}{2} (Xb_w - Ya_w + (1+Z)t_w) + \sin \frac{\theta'}{2} (Xa_w + Yb_w + (1+Z)c_w) \\ \cos \frac{\theta'}{2} (Xc_w + Yt_w + (1+Z)a_w) + \sin \frac{\theta'}{2} (-Xt_w + Yc_w - (1+Z)b_w) \\ \cos \frac{\theta'}{2} (-Xt_w + Yc_w + (1+Z)b_w) + \sin \frac{\theta'}{2} (-Xc_w - Yt_w + (1+Z)a_w) \\ \cos \frac{\theta'}{2} (-Xa_w - Yb_w + (1+Z)c_w) + \sin \frac{\theta'}{2} (Xb_w - Ya_w - (1+Z)t_w) \end{pmatrix}, \quad (13)$$

with

$$\begin{aligned} t_w &= \cos \frac{x}{2} \cos \frac{y}{2} \\ a_w &= \cos \frac{x}{2} \sin \frac{y}{2} \\ b_w &= \cos \frac{y}{2} \sin \frac{x}{2} \\ c_w &= \sin \frac{x}{2} \sin \frac{y}{2} \end{aligned} \quad (14)$$

We can see that the transformation between terrestrial and celestial systems in the form of quaternions is a simple analytical expression, and it can be easily coded.

The corresponding matrix transformation coordinate can be derived from the quaternion components by computing (B16), then taking its transpose. We immediately see that its full analytical development, in contrast to the quaternion form, is very long and largely exceeds the size of an A4 sheet.

3 Validation

In order to validate the quaternion transformation (12) and (13), we reconstruct the corresponding transformation matrix from the quaternion elements through (B16), and compare it with the one calculated from SOFA Fortran programs according to the IAU/IERS standards (IAU 2021).

Here all parameters can be obtained from IERS Conventions models (coded in SOFA library) completed by the Earth Orientation Parameters (EOP) dX , dY , $UT1 - UTC$ or $UT1 - TAI$, x , y , noting that

- $X = X_{\text{mod}} + dX$, $Y = Y_{\text{mod}} + dY$ where “mod” holds for IAU 2006 precession–nutration model.
- $\theta = 2\pi 0.779057273264 + \Omega (UT1 - t_0)$ with $\Omega = 2\pi 1.00273781191135448$ rad/day and t_0 is the First January 2000 at noon in UT1 time scale.

A Fortran program integrates the procedure `iau_C2TXY` to compute the transformation coordinate matrix R between two dates at equidistant steps defined in second. Another

program has been designed in order to calculate the Earth rotation quaternion according its approximate and exact expressions (Eqs. 12 and 13, respectively), and the associated transformation matrix $R_{\tilde{q}}$ and R_q . In both cases, for dates different from 0hUTC, EOP are calculated by interpolating C04 series through cubic splines, modeled celestial coordinates of the CIP are computed according to the SOFA procedure `iau_XY06`, Earth rotation angle θ (incorporating UT1) is calculated through `iau_ERA00`, the position s of the CIO through `iau_S06`, and the position s' of the TIO through `iau_SP00`.

Comparison of R and $R_{\tilde{q}}/R_q$ were carried out over different periods.

In the case of the exact quaternion, the elements of R and R_q present differences below 10^{-14} rad and can be attributed to numerical precision of the computation (double precision in Fortran programmes). An example is displayed in Fig. 1b where R and R_q are calculated between 2019/1/1 at 0hUTC and 2021/1/1 at 0hUTC every 1200 s.

When we consider the approximate quaternion (12), differences increase up to $1.5 \cdot 10^{-12}$ rad for some terms, as shown in Fig. 1a. The observed quasi-annual modulation of the differences mostly results from the neglected term $c_w \approx -xy/4$ rad. But those discrepancies are at least 100 times smaller than the current uncertainty affecting EOP, which is larger than $40 \mu\text{as}$ or 2×10^{-10} rad. Thus, they can be neglected for the astro-geodetic computations. In terms of station position, the approximation impacts the station positions at an insignificant level of 0.01 mm.

4 Full symmetrized transformation

Whereas the CIP is located by Cartesian coordinates (X, Y, Z) in the ICRF, its position in the ITRF is given by elementary rotations of angles x and y . In order to fully symmetrize the form of R , we introduce the spherical coordinates of the CIP in the ITRF, and the associated coordinates x_p and y_p . Let λ_p be the longitude of the CIP in the ITRF, θ_p be the colatitude, the coordinates of the CIP in the ITRF are

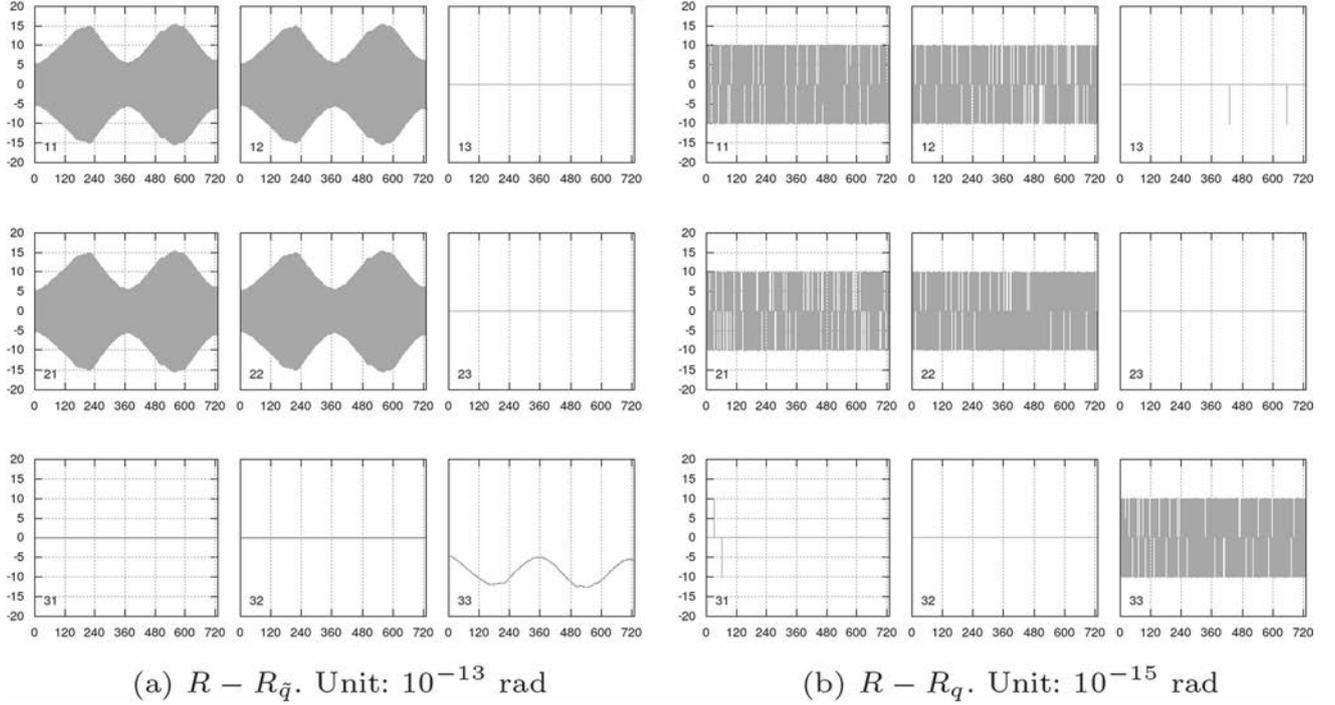


Fig. 1 Differences between Earth rotation matrix elements computed directly from SOFA program and the ones computed from the quaternions, according to the approximate expression (12) (left plot) and exact one (13) (right plot) over 720 days every 1200 s from 2019/1/1 0hUTC

$$\begin{bmatrix} x_p \\ y_p \\ z_p \end{bmatrix} = \begin{bmatrix} \sin \theta_p \cos \lambda_p \\ \sin \theta_p \sin \lambda_p \\ \cos \theta_p \end{bmatrix}, \quad (15)$$

and the transformation is expressed by

$$M = R_3(E)R_2(-d)R_3(-E)R_3(s)R_3(-\theta)R_3(-s'') \\ R_3(-\lambda_p)R_2(\theta_p)R_3(\lambda_p) \quad (16)$$

where the TIO locator s'' is a quantity slightly differing from s' as it will be shown later.

The real pole coordinates x_p and y_p of the CIP in the ITRF can be expressed in function of the angles x and y by applying the coordinate change

$$\begin{bmatrix} x_p \\ y_p \\ z_p \end{bmatrix} = R_1(-y)R_2(-x) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos y & \sin(-y) \\ 1 & \sin y & \cos y \end{bmatrix} \begin{bmatrix} \cos x & 0 & \sin x \\ 0 & 1 & 0 \\ -\sin x & 0 & \cos x \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad (17)$$

that is

$$\begin{bmatrix} x_p \\ y_p \\ z_p \end{bmatrix} = \begin{bmatrix} \cos x & 0 & \sin x \\ \sin x \sin y & \cos y & -\cos x \sin y \\ -\sin x \cos y & \sin y & \cos x \cos y \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\ = \begin{bmatrix} \sin x \\ -\cos x \sin y \\ \cos x \cos y \end{bmatrix}. \quad (18)$$

Now, the quaternion associated with the polar motion transformation $W_p = R_3(-\lambda_p)R_2(\theta_p)R_3(\lambda_p)$ is expressed by

$$q_{W_p} = \begin{bmatrix} \cos \frac{\theta_p}{2} \\ -\sin \frac{\theta_p}{2} \sin \lambda_p \\ \sin \frac{\theta_p}{2} \cos \lambda_p \\ 0 \end{bmatrix}. \quad (19)$$

By analogy to the transformation between (E, d) and (X, Y, Z) according to (6), we have

$$\begin{aligned}\cos \frac{\theta_p}{2} &= \sqrt{\frac{1+z_p}{2}} \\ \sin \frac{\theta_p}{2} \cos \lambda_p &= \frac{x_p}{\sqrt{2(1+z_p)}} \\ \sin \frac{\theta_p}{2} \sin \lambda_p &= \frac{y_p}{\sqrt{2(1+z_p)}}\end{aligned}\quad (20)$$

From that it results

$$q_{W_p} = \frac{1}{\sqrt{2(1+z_p)}} \begin{pmatrix} 1+z_p \\ -y_p \\ x_p \\ 0 \end{pmatrix}. \quad (21)$$

In order to determine the new locator of the TIO s'' in regard to polar rotation matrix W_p , we use the fact that the transformations $R_3(-s'') W_p$ and $R_3(-s') R_2(x) R_1(y)$ have to be equal. This can be expressed by the equality of the quaternions $q_{W_p} \otimes q_3(-s'')$ and $q_1(y) \otimes q_2(x) \otimes q_3(-s')$, leading to $q_{W_p} \otimes q_3(s' - s'') = q_1(y) \otimes q_2(x)$, that is

$$\frac{1}{\sqrt{2(1+z_p)}} \begin{pmatrix} 1+z_p \\ -y_p \\ x_p \\ 0 \end{pmatrix} \begin{pmatrix} \cos \frac{s' - s''}{2} \\ 0 \\ 0 \\ \sin \frac{s' - s''}{2} \end{pmatrix} = \begin{pmatrix} \cos \frac{x}{2} \cos \frac{y}{2} \\ \cos \frac{x}{2} \sin \frac{y}{2} \\ \cos \frac{y}{2} \sin \frac{x}{2} \\ \sin \frac{x}{2} \sin \frac{y}{2} \end{pmatrix} \quad (22)$$

The last component of this relation is written

$$\sqrt{\frac{1+z_p}{2}} \sin \frac{s' - s''}{2} = \sin \frac{x}{2} \sin \frac{y}{2}, \quad (23)$$

that is, after neglecting terms of order higher than or equal to 3

$$s'' - s' \approx -\frac{xy}{\sqrt{2(1+z_p)}} \approx -\frac{xy}{2}, \quad (24)$$

which indicates a difference of about 10^{-12} radians. A matrix calculation allows a geometrical interpretation of the difference $s'' - s'$ as exposed in Appendix C.

Actually, the TIO locator is expressed by definition as a function of the pole coordinates (x_p, y_p) (Capitaine et al. 1986), but in the IERS convention (Petit and Luzum 2010) (x_p, y_p) are approximated by the two rotation angles x and $-y$, respectively. Using the true pole coordinates as introduced above is strictly consistent with the concept of the TIO locator.

The total transformation is represented by the quaternion $q = q_{W_p} \otimes q_3(-s'' - \theta + s) \otimes q_{PN}$, that is

$$q_M = \frac{1}{\sqrt{2(1+z_p)}} \frac{1}{\sqrt{2(1+Z)}} \begin{pmatrix} 1+z_p \\ -y_p \\ x_p \\ 0 \end{pmatrix} \begin{pmatrix} \cos \frac{-s'' - \theta + s}{2} \\ 0 \\ 0 \\ \sin \frac{-s'' - \theta + s}{2} \end{pmatrix} \begin{pmatrix} 1+Z \\ Y \\ -X \\ 0 \end{pmatrix}. \quad (25)$$

This gives the quaternion as a function of the direction cosines of the CIP in both GCRS and ITRS:

$$q = \frac{1}{2\sqrt{(1+z_p)(1+Z)}} \begin{pmatrix} \cos \frac{\theta''}{2} [Xx_p + Yy_p + (1+Z)(1+z_p)] - \sin \frac{\theta''}{2} [-Xy_p + Yx_p] \\ \cos \frac{\theta''}{2} [Y(1+z_p) - (1+Z)y_p] + \sin \frac{\theta''}{2} [-X(1+z_p) + (1+Z)x_p] \\ \cos \frac{\theta''}{2} [-X(1+z_p) + (1+Z)x_p] + \sin \frac{\theta''}{2} [-Y(1+z_p) - (1+Z)y_p] \\ \cos \frac{\theta''}{2} [Xy_p - Yx_p] + \sin \frac{\theta''}{2} [Xx_p + Yy_p - (1+Z)(1+z_p)] \end{pmatrix}, \quad (26)$$

where $\theta'' = \theta + s'' - s$.

5 Conclusion

Quite absent from Earth rotation studies where Euler angles or matrices are favored, the quaternions are a pertinent alternative for describing the Earth rotation in compliance with the parametrization based upon the direction cosines of the CIP (recommended by the IAU). Indeed, the rotation quaternion between celestial and terrestrial reference systems presents a simple analytical expression, that is impossible to obtain in the case of the matrix formalism. We check its coherence with the matrix calculation of SOFA Fortran library of the IAU to within 10^{-12} rad for its approximate form and 10^{-14} rad for its exact one.

The parametrization of the CIP was extended to its direction cosines in the ITRF, replacing the common pole coordinates and ending up in a full symmetrized expression of the Earth rotation quaternion.

Various tests showed that the expression (13) or (26) do not produce a significant CPU time gain with respect to the direct

numerical calculation of the corresponding matrix. But, due to their short analytical form, those expressions can be easily coded in any software requiring the Earth rotation transformation (telescope pointing, reduction of astro-geodetic data). Actually, a precision of $1.5 \cdot 10^{-12}$ rad ($0.3 \mu\text{as}$) is sufficient for many applications, and the approximate version (12) can then be used. Moreover, in the context of the astro-geodetic reduction, those expressions allow to easily derive the partial derivatives of the measured quantities with respect to the EOP $X, Y, \text{UT1}, (x, y)$ or (x_p, y_p) .

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Author Contributions CB designed research and wrote the paper. YC proposed Sect. 4 and made an independent check of the computations.

Appendices

A Definition of the quaternions and properties

A.1 Definition

The *quaternions* were discovered by the Irish Mathematician William Rowan Hamilton in the early 1840's (Hamilton 1866). The problem was to find a multiplication law \otimes for n -uplets of real number $X = [x_1, x_2, \dots, x_n]$ such that the modulus of a product $X \otimes X'$ is equal to the product of the modulus $|X||X'|$:

$$|X||X'| = |X \otimes X'| \quad (\text{A1})$$

the modulus being defined by $|X| = \sqrt{X\bar{X}}$ where $\bar{X} = [x_1, -x_2, -x_3, \dots]$ is the conjugate.

Numbers obeying such a law constitute a generalization of complex numbers. Hamilton had already searched for a solution to this problem for the tri-uplet, but unsuccessfully. Considering then a fourth dimension, he obtained a multiplication law, which satisfies Eq.(A1). Considering two quadruplets $q_1 = [t_1, x_1, y_1, z_1]$ and $q_2 = [t_2, x_2, y_2, z_2]$, their multiplication is given by

$$q_1 \otimes q_2 = \begin{bmatrix} t_1 t_2 - x_1 x_2 - y_1 y_2 - z_1 z_2 \\ t_1 x_2 + x_1 t_2 + y_1 z_2 - z_1 y_2 \\ t_1 y_2 - x_1 z_2 + y_1 t_2 + z_1 x_2 \\ t_1 z_2 + x_1 y_2 - y_1 x_2 + z_1 t_2 \end{bmatrix} \quad (\text{A2})$$

Endowed with this internal composition, these quadruplets of real numbers $q = [t, x, y, z]$ are called quaternions and have a structure of group.

The addition of two quaternions is defined by

$$q_1 + q_2 = [t_1 + t_2, x_1 + x_2, y_1 + y_2, z_1 + z_2]. \quad (\text{A3})$$

It can be easily shown that the multiplication law is associative and distributive with respect to the addition. However, it is not commutative. Let be $i = [0, 1, 0, 0]$, $j = [0, 0, 1, 0]$, $k = [0, 0, 0, 1]$, any quaternion $[t, x, y, z]$ can be expressed by

$$q = t + ix + jy + kz. \quad (\text{A4})$$

Any quaternion of the form $(t, 0, 0, 0)$ is reduced to the real number t . From the multiplication law (A2) i, j and k check the properties

$$\begin{aligned} i \otimes j &= -j \otimes i = k, & j \otimes k &= -k \otimes j = i, \\ k \otimes i &= -i \otimes k = j, \\ i^2 &= i \otimes i = j^2 = k^2 = ijk = [-1, 0, 0, 0] \end{aligned} \quad (\text{A5})$$

These relations determine fully the multiplication of two quaternions.

A.2 Scalar and vectorial parts

We can consider (x, y, z) as the components of a vector \vec{V} in a direct orthonormal basis, so that every quaternion can be expressed by

$$q = [t, \vec{V}], \quad (\text{A6})$$

where \vec{V} is the *vectorial part* of the quaternion and t is its *scalar part*. As the vectorial part is referred to a direct orthonormal basis, (A2) can be rewritten

$$\begin{aligned} q_1 \otimes q_2 &= [t_1, \vec{V}_1] \otimes [t_2, \vec{V}_2] = q_3 = [t_3, \vec{V}_3] \\ &\text{with } t_3 = t_1 t_2 - \vec{V}_1 \cdot \vec{V}_2 \text{ and} \\ &\vec{V}_3 = t_2 \vec{V}_1 + t_1 \vec{V}_2 + \vec{V}_1 \wedge \vec{V}_2, \end{aligned} \quad (\text{A7})$$

where $\vec{V}_1 \cdot \vec{V}_2$ is the scalar product of the vectorial parts and $\vec{V}_1 \wedge \vec{V}_2$ their vectorial product.

The non-commutativity of the product of two quaternions comes from the non-commutativity of the vectorial product $\vec{V}_1 \wedge \vec{V}_2$.

A.3 Symmetric and antisymmetric parts

According to (A7), the symmetric part of the product of two quaternions is

$$\begin{aligned} \langle q_1 \otimes q_2 \rangle &= \frac{1}{2}(q_1 \otimes q_2 + q_2 \otimes q_1) \\ &= [t_1 t_2 - \vec{V}_1 \cdot \vec{V}_2, t_2 \vec{V}_1 + t_1 \vec{V}_2], \end{aligned} \quad (\text{A8})$$

and the antisymmetric part

$$[q_1 \otimes q_2] = \frac{1}{2}(q_1 \otimes q_2 - q_2 \otimes q_1) = [0, \vec{V}_1 \wedge \vec{V}_2]. \quad (\text{A9})$$

A.4 Conjugate and modulus

The *conjugate* of a quaternion $q = [t, \vec{V}]$ is defined by $\bar{q} = [t, -\vec{V}]$. Then, it can be easily shown that the conjugate of a product of two quaternions is the inverse product of the conjugates:

$$\overline{q_1 \otimes q_2} = \bar{q}_2 \otimes \bar{q}_1. \quad (\text{A10})$$

The *norm* or the *modulus* of a quaternion q is

$$|q| = \sqrt{q \otimes \bar{q}} = \sqrt{\bar{q} \otimes q} = \sqrt{t^2 + x^2 + y^2 + z^2}. \quad (\text{A11})$$

A.5 Inverse

From A11, any non-zero quaternion q has an inverse, written q^{-1} , such as $q \otimes q^{-1} = q^{-1} \otimes q = (1, 0, 0, 0)$ and given by

$$q^{-1} = \frac{\bar{q}}{|q|^2}. \quad (\text{A12})$$

A.6 An useful lemme

Let $q = (t, \vec{\Gamma})$ be any quaternion and $(0, \vec{V})$ a pure vectorial quaternion, the following expressions can be easily derived:

$$\begin{aligned} q \otimes (0, \vec{V}) \otimes \bar{q} \\ = \left[0, 2(\vec{\Gamma} \cdot \vec{V})\vec{\Gamma} + (t^2 - \vec{\Gamma}^2)\vec{V} + 2t\vec{\Gamma} \wedge \vec{V} \right], \end{aligned} \quad (\text{A13})$$

and

$$\begin{aligned} \bar{q} \otimes [0, \vec{V}] \otimes q \\ = \left[0, 2(\vec{\Gamma} \cdot \vec{V})\vec{\Gamma} + (t^2 - \vec{\Gamma}^2)\vec{V} - 2t\vec{\Gamma} \wedge \vec{V} \right]. \end{aligned} \quad (\text{A14})$$

B Representation of a rotation by a unit quaternion

B.1 Quaternion of rotation

Any rotation can be fully defined by the rotation angle ϕ and the direction of its rotation axis, characterized by its direction cosines $(\alpha, \beta, \gamma) = \hat{u}$ in a direct orthonormal basis. From a given vector \vec{r} and the vector \hat{u} , we can build another orthonormal direct basis $(\hat{u}, \hat{v}, \hat{w})$ such as

$$\begin{aligned} \hat{w} &= \frac{\hat{u} \wedge \vec{r}}{\|\hat{u} \wedge \vec{r}\|} \\ \hat{v} &= \hat{w} \wedge \hat{u} = \frac{\vec{r} - (\hat{u} \cdot \vec{r})\hat{u}}{\|\hat{u} \wedge \vec{r}\|} \end{aligned} \quad (\text{B1})$$

Let us apply the rotation of angle ϕ and direction vector \hat{u} on the vector \vec{r} , which becomes \vec{r}' . The projection of \vec{r}' on the rotation axis \hat{u} is unchanged and equal to $\vec{r} \cdot \hat{u}$. Its second component along \hat{v} , namely $\vec{r} \cdot \hat{v}$, is rotated by the angle ϕ in the plane (\hat{v}, \hat{w}) . Its third component, along \hat{w} is equal to zero, for \hat{w} is perpendicular to \vec{r} . It results

$$\vec{r}' = (\vec{r} \cdot \hat{u})\hat{u} + (\vec{r} \cdot \hat{v})\cos\phi\hat{v} + (\vec{r} \cdot \hat{w})\sin\phi\hat{w}. \quad (\text{B2})$$

By using the last expression of (B1), we get $\hat{v} \cdot \hat{v} = 1 = \vec{r} \cdot \hat{v} / \|\hat{u} \wedge \vec{r}\|$, that is $\vec{r} \cdot \hat{v} = \|\hat{u} \wedge \vec{r}\|$, allowing to derive $\hat{v} = [\vec{r} - (\hat{u} \cdot \vec{r})\hat{u}] / \vec{r} \cdot \hat{v}$ and $\hat{w} = \hat{u} \wedge \vec{r} / \vec{r} \cdot \hat{v}$, and finally

$$\begin{aligned} \vec{r}' &= (1 - \cos\phi)(\hat{u} \cdot \vec{r})\hat{u} \\ &\quad + \cos\phi\vec{r} + \sin\phi\hat{u} \wedge \vec{r}. \end{aligned} \quad (\text{B3})$$

On the other hand, applying the transformation (A13) to the quaternion vector $[0, \vec{r}']$ and quaternion $q = (t, \vec{\Gamma})$, yields

$$\begin{aligned} q \otimes [0, \vec{r}'] \otimes \bar{q} &= [0, \vec{r}'] \\ &= \left[0, 2(\vec{\Gamma} \cdot \vec{r}')\vec{\Gamma} + (t^2 - \vec{\Gamma}^2)\vec{r}' + 2t\vec{\Gamma} \wedge \vec{r}' \right]. \end{aligned} \quad (\text{B4})$$

So, the quaternion $q = (t, \vec{\Gamma})$ corresponds to the rotation transformation (B3) if it obeys the system

$$\begin{aligned} 2(\vec{\Gamma} \cdot \vec{r}')\vec{\Gamma} &= (1 - \cos\phi)(\hat{u} \cdot \vec{r}')\hat{u} \\ t^2 - \vec{\Gamma}^2 &= \cos\phi \\ 2t\vec{\Gamma} \wedge \vec{r}' &= \sin\phi\hat{u} \wedge \vec{r}' \end{aligned}, \quad (\text{B5})$$

with solutions

$$q = (t, \vec{\Gamma}) = \pm \left[\cos\left(\frac{\phi}{2}\right), \sin\left(\frac{\phi}{2}\right)\hat{u} \right]. \quad (\text{B6})$$

By definition, the *quaternion of rotation* $q_{(\phi, \hat{u})}$ associated with the rotation (ϕ, \hat{u}) is

$$q_{(\phi, \hat{u})} = \left[\cos\left(\frac{\phi}{2}\right), \sin\left(\frac{\phi}{2}\right) \hat{u} \right]. \quad (\text{B7})$$

The rotation of same axis but of opposite angle is given by the conjugate quaternion $\bar{q}_{(\phi, \hat{u})}$.

Both these quaternions allows to obtain the transform vector \vec{r}' according to

$$[0, \vec{r}'] = q_{(\phi, \hat{u})} \otimes [0, \vec{r}] \otimes \bar{q}_{(\phi, \hat{u})}. \quad (\text{B8})$$

B.2 Coordinate transformation

Let (x, y, z) be the coordinates of a given vector \vec{r} in the Cartesian frame Oxyz endowed with the orthonormal basis $(\hat{i}, \hat{j}, \hat{k})$. By applying the quaternion of rotation $q_{(\phi, \hat{u})}$ on $(\hat{i}, \hat{j}, \hat{k})$, this one is transformed into the basis $(\hat{i}', \hat{j}', \hat{k}')$. Let (x', y', z') be the coordinate of \vec{r} in this new basis. Then, two equal vectorial quaternions give \vec{r} :

$$x'[0, \hat{i}'] + y'[0, \hat{j}'] + z'[0, \hat{k}'] = x[0, \hat{i}] + y[0, \hat{j}] + z[0, \hat{k}]. \quad (\text{B9})$$

According to the transformation law (B8), this relation becomes

$$\begin{aligned} & x'q_{(\phi, \hat{u})} \otimes [0, \hat{i}'] \otimes \bar{q}_{(\phi, \hat{u})} + y'q_{(\phi, \hat{u})} \otimes [0, \hat{j}'] \otimes \bar{q}_{(\phi, \hat{u})} \\ & + z'q_{(\phi, \hat{u})} \otimes [0, \hat{k}'] \otimes \bar{q}_{(\phi, \hat{u})} \\ & = x[0, \hat{i}] + y[0, \hat{j}] + z[0, \hat{k}]. \end{aligned} \quad (\text{B10})$$

Then

$$\begin{aligned} & \bar{q}_{(\phi, \hat{u})} \otimes q_{(\phi, \hat{u})} \left(x'[0, \hat{i}'] + y'[0, \hat{j}'] + z'[0, \hat{k}'] \right) \\ & \bar{q}_{(\phi, \hat{u})} \otimes q_{(\phi, \hat{u})} = \bar{q}_{(\phi, \hat{u})} \otimes \left(x[0, \hat{i}] + y[0, \hat{j}] + z[0, \hat{k}] \right) \\ & \quad \otimes q_{(\phi, \hat{u})}. \end{aligned} \quad (\text{B11})$$

As $\bar{q}_{(\phi, \hat{u})} \otimes q_{(\phi, \hat{u})} = \bar{q}_{(\phi, \hat{u})} \otimes q_{(\phi, \hat{u})} = |q_{(\phi, \hat{u})}|^2 = 1$, we obtain

$$\begin{aligned} & x'[0, \hat{i}'] + y'[0, \hat{j}'] + z'[0, \hat{k}'] \\ & = \bar{q}_{(\phi, \hat{u})} \otimes \left(x[0, \hat{i}] + y[0, \hat{j}] + z[0, \hat{k}] \right) \otimes q_{(\phi, \hat{u})}. \end{aligned} \quad (\text{B12})$$

or

$$[0, x', y', z'] = \bar{q}_{(\phi, \hat{u})} \otimes [0, x, y, z] \otimes q_{(\phi, \hat{u})}. \quad (\text{B13})$$

Noting $\underline{x} = (x, y, z)$ and $\underline{x}' = (x', y', z')$, the former relation can be expressed by

$$[0, \underline{x}'] = \bar{q}_{(\phi, \hat{u})} \otimes [0, \underline{x}] \otimes q_{(\phi, \hat{u})}. \quad (\text{B14})$$

This is the *passive transformation* in the sense that it only transforms the vector coordinates and not the vector itself, in contrast to the *active transformation* (B8).

B.3 Equivalent matrix for coordinate transformation

Equation (B3) allows to calculate the components of the new basis in the old one, i.e., the transformation matrix P from the basis $(\hat{i}, \hat{j}, \hat{k})$ to the basis $(\hat{i}', \hat{j}', \hat{k}')$:

$$\begin{aligned} P &= (1 - \cos \phi) \begin{bmatrix} \alpha^2 & \beta\alpha & \gamma\alpha \\ \alpha\beta & \beta^2 & \gamma\beta \\ \alpha\gamma & \beta\gamma & \gamma^2 \end{bmatrix} + \cos \phi I \\ &+ \sin \phi \begin{bmatrix} 0 & -\gamma & \beta \\ \gamma & 0 & -\alpha \\ -\beta & \alpha & 0 \end{bmatrix}, \end{aligned} \quad (\text{B15})$$

where α , β and γ are the direction cosines of \hat{u} .

Considering the components $t = \cos \phi/2$, $a = \alpha \sin \phi/2$, $b = \beta \sin \phi/2$, $c = \gamma \sin \phi/2$ of the quaternion $q_{(\phi, \hat{u})} = [t, a, b, c]$, P takes the form

$$P = \begin{bmatrix} t^2 + a^2 - b^2 - c^2 & 2(ab - ct) & 2(ac + bt) \\ 2(ab + ct) & t^2 - a^2 + b^2 - c^2 & 2(bc - at) \\ 2(ac - bt) & 2(bc + at) & t^2 - a^2 - b^2 + c^2 \end{bmatrix}. \quad (\text{B16})$$

Then, the coordinate transformation can be written as

$$\underline{x}' = P^T \underline{x} = R \underline{x}, \quad (\text{B17})$$

where the coordinate transformation using the matrix R is equivalent to that using the quaternion q given by (B14).

B.4 Quaternion representing the product of two rotations

Let q_1 and q_2 be the quaternions pertaining to the rotations R_1 and R_2 , respectively (with secant axes). From (B8), the active sequence of rotations $R_2 R_1$ (first R_1 then R_2) applied to the vector \vec{r} gives

$$\begin{aligned} [0, \vec{r}'] &= (q_2 \otimes q_1) \otimes [0, \vec{r}] \otimes (\bar{q}_1 \otimes \bar{q}_2) \\ &= (q_2 \otimes q_1) \otimes [0, \vec{r}] \otimes \overline{q_2 \otimes q_1}. \end{aligned} \quad (\text{B18})$$

According to (B14), the passive sequence transforming the coordinates \underline{x} of \vec{r} to \underline{x}' is

$$\begin{aligned} [0, \underline{x}'] &= (\bar{q}_2 \otimes \bar{q}_1) \otimes [0, \underline{x}] \otimes (q_1 \otimes q_2) \\ &= \overline{q_1 \otimes q_2} \otimes [0, \underline{x}] \otimes (q_1 \otimes q_2). \end{aligned} \quad (\text{B19})$$

So, the product $R_2 R_1$ is associated with the quaternion $q_2 \otimes q_1$ in the case of the active transformation and to $q_1 \otimes q_2$ in the case of the passive transformation.

C Matrix-based calculation of $s' - s''$

According to (23), we derive

$$\begin{aligned} \cos(s' - s'') &= 1 - 2 \frac{2}{1 + z_p} \sin^2 \frac{x}{2} \\ \sin^2 \frac{y}{2} &= \frac{\cos x + \cos y}{1 + z_p}. \end{aligned} \quad (\text{C1})$$

This difference actually comes from a residual rotation between the 2-rotation sequence $R_2(x)R_1(y)$ and the 3-rotation sequence $W_p = R_3(-\lambda_p)R_2(\theta_p)R_3(\lambda_p)$. To see this, let us apply those rotation sequences on the unit vector \hat{x} of the ITRS basis ($\hat{x}, \hat{y}, \hat{z}$). The resulting vector in the ITRS basis is the first column of the transpose of the corresponding transformation coordinate matrix.

So, for the transformation $R_2(x)R_1(y)$, we have to consider the matrix

$$R_1(-y)R_2(-x) = \begin{bmatrix} \cos x & 0 & \sin x \\ \sin x \sin y & \cos y & -\cos x \sin y \\ -\sin x \cos y & \sin y & \cos x \cos y \end{bmatrix} \quad (\text{C2})$$

of which the first column gives the vector

$$\hat{x}_1 = \cos x \hat{x} + \sin x \sin y \hat{y} - \sin x \cos y \hat{z}. \quad (\text{C3})$$

For the 3 rotation sequence W_p , it should be noticed that the transformation coordinate $W_p^T R_3(s'')$ from the TIRS to the ITRS is similar to $PN(X, Y)R_3(s)$ going from the CIRS to the GCRS. So W_p^T can be easily written by replacing X with x_p and Y with y_p in the conventional precession–nutation matrix $PN(X, Y)$ given by (2), taking as well $a = 1/(1 + z_p) = 1/(1 + \cos x \cos y)$:

$$\begin{aligned} W_p^T &= R_3(\lambda_p)R_2(-\theta_p)R_3(-\lambda_p) \\ &= \begin{bmatrix} 1 - ax_p^2 & -ax_p y_p & x_p \\ -ax_p y_p & 1 - ay_p^2 & y_p \\ -x_p & -y_p & 1 - a(x_p^2 + y_p^2) \end{bmatrix}. \end{aligned} \quad (\text{C4})$$

The corresponding transformed vector is

$$\hat{x}_2 = (1 - ax_p^2) \hat{x} - ax_p y_p \hat{y} - x_p \hat{z} \quad (\text{C5})$$

that is, from (18),

$$\begin{aligned} \hat{x}_2 &= \left(1 - \frac{\sin^2 x}{1 + \cos x \cos y} \right) \\ &\hat{x} + \frac{\sin x \cos x \sin y}{1 + \cos x \cos y} \hat{y} - \sin x \hat{z}. \end{aligned} \quad (\text{C6})$$

Then the angle between the vectors \hat{x}_1 and \hat{x}_2 is precisely the difference $s'' - s'$ with the cosine

$$\begin{aligned} \cos(s'' - s') &= \vec{x}_1 \cdot \vec{x}_2 \\ &= \frac{1}{1 + z_p} (\cos x + \cos^2 x \cos y \\ &\quad - \cos x \sin^2 x + \sin^2 x \cos x \sin^2 y \\ &\quad + \sin^2 x \cos y + \sin^2 x \cos x \cos^2 y) \\ &= \frac{\cos x + \cos y}{1 + z_p}, \end{aligned} \quad (\text{C7})$$

which is exactly the expression (C1) that we found by comparing quaternion expressions.

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